Notes for Lecture 27

Wed, 10/30/2024

Quiz. Four limits that we can "see" plus one that we need to work out, like the following:

$$\lim_{n\to\infty}\sqrt[n]{\frac{3}{n^2}} = \lim_{n\to\infty}\left(\frac{3}{n^2}\right)^{1/n} = \lim_{n\to\infty}\exp\left(\ln\left(\left(\frac{3}{n^2}\right)^{1/n}\right)\right) = \lim_{n\to\infty}\exp\left(\frac{1}{n}\ln\left(\frac{3}{n^2}\right)\right) = \exp(0) = 1$$

where we used that

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{3}{n^2}\right) = \lim_{n\to\infty}\frac{\ln(3) - 2\ln(n)}{n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\underset{\text{LH}}{=}} \lim_{n\to\infty}\frac{-2\cdot\frac{1}{n}}{1} = 0.$$

Series

A tortoise racing a Greek hero... **Zeno's paradox**:

https://en.wikipedia.org/wiki/Zeno%27s_paradoxes#Achilles_and_the_tortoise

Example 127. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \sum_{n=1}^{\infty}\frac{1}{2^n} = 1$

Solution. Visual!

Solution. Redo this example by taking the limit of a geometric sum.

Geometric series

Review. The geometric sum is

$$\sum_{n=0}^{M} x^n = 1 + x + x^2 + \ldots + x^M = \frac{1 - x^{M+1}}{1 - x}.$$

Taking the limit $M \to \infty$ in the geometric sum, we get: (recall that $\lim_{M\to\infty} x^M = 0$ if $|x| < 1$)

(geometric series) If $|x| < 1$, then

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1 - x}.$$

If $|x| \geqslant 1$, then the geometric series diverges.

Example 128. Compute the following series (or state that it diverges):

(a) $\sum_{n=0}^{\infty}\frac{1}{2^n}$

(b) $\sum_{n=3}^{\infty}\frac{1}{2^n}$

(c) $\sum_{n=0}^{\infty}\frac{7}{10^n}$

(d) $\sum_{n=2}^{\infty}\frac{7}{10^n}$

(e) $\sum_{n=0}^{\infty}\frac{5}{3^n}$

(f) $\sum_{n=2}^{\infty} 3\cdot 4^{-n}$

(g) $\sum_{n=0}^{\infty}\left(\frac{7}{2^n} - \frac{3^n}{5^n}\right)$

(h) $\sum_{n=0}^{\infty}\frac{5^n}{3^n}$

(i) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

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Solution.

(a) $\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

(b) $\sum_{n=3}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2}\right) = 2 - \left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{4}$

(c) $\sum_{n=0}^{\infty} \frac{7}{10^n} = 7\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 7 \cdot \frac{1}{1-\frac{1}{10}} = \frac{70}{9}$

(d) $\sum_{n=2}^{\infty} \frac{7}{10^n} = \sum_{n=0}^{\infty} \frac{7}{10^n} - \left(\frac{7}{10^0} + \frac{7}{10^1}\right) = \frac{70}{9} - \left(7 + \frac{7}{10}\right) = \frac{7}{90}$

(e) $\sum_{n=0}^{\infty} \frac{5}{3^n} = 5\sum_{n=0}^{\infty} \frac{1}{3^n} = 5 \cdot \frac{1}{1-\frac{1}{3}} = \frac{15}{2}$

(f) $\sum_{n=2}^{\infty} 3 \cdot 4^{-n} = 3\sum_{n=2}^{\infty} \frac{1}{4^n} = 3\left(\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4}\right) = \frac{1}{4}$

(g) $\sum_{n=0}^{\infty} \left(\frac{7}{2^n} - \frac{3^n}{5^n}\right) = 7\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 7 \cdot \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{3}{5}} = 14 - \frac{5}{2} = \frac{23}{2}$

(h) $\sum_{n=0}^{\infty} \frac{5^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n$ doesn't converge because $\left|\frac{5}{3}\right| \geqslant 1$.

(i) $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ provided that $|-x^2| < 1$ (which is the same as $|x| < 1$). If this condition is not true, then the series diverges.

The very last example illustrates an important point. Namely, it shows that there is a novel way to think about (and get our hands on) functions like $\frac{1}{1+x^2}$.

Recall that we care about this function in particular, because it was a building block in partial fractions. For instance, we know that its antiderivative is $\arctan(x)$.

This is the main reason why we are learning about series in a course that focuses on functions!

We will see that it is very convenient to work with series representing functions: they can be differentiated and integrated, and give us an opportunity to work with functions that cannot be written in terms of the "usual" functions.

Example 129. Express the number $0.7777\ldots$ as a rational number.

Solution. (using geometric series)

$$0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots = \sum_{n=1}^{\infty} \frac{7}{10^n} = 7\left(\frac{1}{1-\frac{1}{10}} - 10^0\right) = 7\left(\frac{10}{9} - 1\right) = \frac{7}{9}$$

Solution. (highschool) Everyone is familiar with $0.3333\ldots = \frac{1}{3}$. This implies that $0.1111\ldots = \frac{1}{3} \cdot 0.3333\ldots = \frac{1}{9}$. Hence, our number is $0.7777\ldots = 7 \cdot 0.1111\ldots = \frac{7}{9}$.

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