Example 130. (Halloween scare!) Let $a=b$. Then $a^2=ab$, so $a^2+a^2=a^2+ab$ or $2a^2=a^2+ab$. Hence, $2a^2-2ab=a^2-ab$ or $2(a^2-ab)=a^2-ab$. Cancelling, we arrive at $2=1$.

[Can you see the foul but disguised division by zero?!]

Example 131. Explain the following:

$$\frac{1}{499}=0.002004008016032064128256\ldots$$

Solution. The thing to explain is that there appear to be powers of 2 in the decimal on the right-hand side. Note that the right-hand side can be realized as a geometric series:

$$\frac{2}{1000}+\left(\frac{2}{1000}\right)^2+\left(\frac{2}{1000}\right)^3+\ldots=\frac{1}{1-\frac{2}{1000}}-1=\frac{1}{499}.$$

Example 132. (Halloween scare!) Foul play with divergent series:

$$\begin{aligned} 0 &= (1-1)+(1-1)+(1-1)+\ldots \\ &= 1-1+1-1+1-1+\ldots \\ &= 1+(-1+1)+(-1+1)+\ldots \\ &= 1+0+0\ldots \\ &= 1 \end{aligned}$$

Where did this go wrong? Note that in the second line we have the series $\sum_{n=0}^{\infty}(-1)^n$ which is divergent.

Lesson. Divergent series don't conform to our usual laws.

In other words, convergence of series is crucially important when working with them.

The fallacy above is somewhat similar to the argument "$\infty=\infty+1$, so $0=1$".

Example 133. (geometric series, again) To follow-up the previous example on a positive note: if a series converges, then we can work with it. The following argument for evaluating the geometric series is valid provided that the series converges (which we know it is if $|x|<1$):

$$S=1+x+x^2+x^3+\ldots \quad \rightsquigarrow \quad xS=x+x^2+x^3+x^4+\ldots=S-1 \quad \rightsquigarrow \quad S=\frac{1}{1-x}$$

Example 134. Write the series $e^{2x}+e^{3x}+e^{4x}+\ldots$ using Σ-notation and evaluate it.

Solution. $e^{2x}+e^{3x}+e^{4x}+\ldots=\sum_{n=2}^{\infty}e^{nx}=\sum_{n=0}^{\infty}e^{nx}-1-e^x=\frac{1}{1-e^x}-1-e^x$

Here, we need to assume that $|e^x|<1$ (or, equivalently, $x<0$).

Alternatively. $e^{2x}+e^{3x}+e^{4x}+\ldots=e^{2x}(1+e^x+e^{2x}+\ldots)=e^{2x}\sum_{n=0}^{\infty}e^{nx}=\frac{e^{2x}}{1-e^x}$

Check that this is the same!

Armin Straub
straub@southalabama.edu

Example 135. Express the number $2.313131...$ as a rational number.

Solution.

$$2.313131... = 2 + \frac{31}{100} + \frac{31}{(100)^2} + \frac{31}{(100)^3} + ... = 2 + 31\sum_{n=1}^{\infty} \frac{1}{100^n} = 2 + 31\left(\frac{1}{1-\frac{1}{100}} - \frac{1}{100^0}\right) = 2 + \frac{31}{99} = \frac{229}{99}$$

This example plus the last one from previous class teach us something fundamental about numbers:

> Rational numbers are precisely those numbers which have a finite (like 1.5) or repeating (like $2.313131...$) decimal expansion.
>
> Moreover, there is some ambiguity because finite decimals, like 1.5, can also be written in the repeating fashion $1.5 = 1.4999....$

As a consequence, irrational numbers like $\sqrt{2}$ or π never have a repeating decimal expansion.

The nth-term test for divergence

Review. Recall that the improper integral $\int_N^{\infty} f(x)\mathrm{d}x$ converges if and only if the limit $\lim_{M\to\infty} \int_N^M f(x)\mathrm{d}x$ exists.
Likewise, $\sum_{n=N}^{\infty} a_n$ converges if and only if the limit $\lim_{M\to\infty} \sum_{n=N}^{M} a_n$ exists.

For the series $\sum_{n=N}^{\infty} a_n$ to converge, it is necessary that $\lim_{n\to\infty} a_n = 0$.

Can you explain how this follows from the definition of limit?

Intuitively, this is simply saying that the only hope to be able to add infinitely many things (and get something finite) is if these things are very small.

> **Theorem 136. (nth-term test for divergence)** If $\lim_{n\to\infty} a_n$ is not 0, then $\sum_{n=1}^{\infty} a_n$ diverges.

(In particular, if the limit $\lim_{n\to\infty} a_n$ does not exist, then the series diverges.)

Example 137. Show that the following series all diverge.

(a) $\sum_{n=0}^{\infty} \frac{3^n + 5^n}{10 \cdot 5^n}$ (b) $\sum_{n=1}^{\infty} (-1)^n$ (c) $\sum_{n=1}^{\infty} \frac{n^2}{3n^2+7}$ (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\log(n)}$

Solution.

(a) Note that $\lim_{n\to\infty} \frac{3^n + 5^n}{10 \cdot 5^n} = \frac{1}{10} \neq 0$. Hence, the series diverges by the nth-term test for divergence.

(b) The sequence $(-1)^n$ does not converge to 0 as $n \to \infty$. Hence, the series diverges by the nth-term test.

(c) Since $\lim_{n\to\infty} \frac{n^2}{3n^2+7} = \frac{1}{3} \neq 0$ the series diverges by the nth-term test for divergence.

(d) This series diverges by the nth-term test for divergence because $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n}$ is not zero.
In fact, $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \infty$. (If you don't see this, apply L'Hospital!)

A word of caution. The nth-term test for divergence only gives a necessary condition. It is not sufficient!

For instance, as we will see next time, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges although $\lim_{n\to\infty} \frac{1}{n} = 0$.