**Example 155.** Determine the radius of convergence of the following power series and their exact interval of convergence.

(a) $\sum_{n=0}^{\infty} (n^2+4)\, x^n$ — This is a power series about $x=0$.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} (x-3)^n$ — This is a power series about $x=3$.

**Solution.**

(a) We apply the ratio test with $a_n = (n^2+4)\, x^n$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)^2+4)\, x^{n+1}}{(n^2+4)\, x^n}\right| = |x| \frac{n^2+2n+5}{n^2+4} \to |x|$ as $n \to \infty$

The ratio test implies that $\sum_{n=0}^{\infty} (n^2+4)\, x^n$ converges if $|x| < 1$.

Thus the radius of convergence is $1$.

The ratio test does not tell us what happens when $|x|=1$. We now look at those cases more carefully:

- $x=1$: $\sum_{n=0}^{\infty} (n^2+4)$ clearly diverges (because $\lim_{n\to\infty} (n^2+4)$ is not $0$).
- $x=-1$: $\sum_{n=0}^{\infty} (n^2+4)(-1)^n$ clearly diverges (because $\lim_{n\to\infty} (n^2+4)(-1)^n$ is not $0$).

Combined, $\sum_{n=0}^{\infty} (n^2+4)\, x^n$ converges if and only if $x$ is in $(-1,1)$ (the exact interval of convergence).

(b) We apply the ratio test with $a_n = \frac{2^n}{\sqrt{n}}(x-3)^n$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{2^n(x-3)^n}\right| = 2|x-3| \sqrt{\frac{n}{n+1}} \to 2|x-3|$ as $n \to \infty$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-3)^n$ converges if $|x-3| < \frac{1}{2}$.

So the radius of convergence is $\frac{1}{2}$.

The ratio test is inconclusive for $|x-3| = \frac{1}{2}$ or, equivalently, $x = 3 - \frac{1}{2} = \frac{5}{2}$ and $x = 3 + \frac{1}{2} = \frac{7}{2}$:

- $x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test ( $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$).
  **Comment.** We could have also just recognized this as the alternating $p$-series with $p = \frac{1}{2}$.
- $x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is the $p$-series with $p = \frac{1}{2}$ which diverges (because $p \leqslant 1$).

Combined, the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-3)^n$ is $\left[\frac{5}{2}, \frac{7}{2}\right)$.

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**Example 156.** Determine the radius of convergence of $\sum_{n=1}^{\infty} \frac{5^n}{n^2}(4x-3)^{2n}$ (this is a power series about $\frac{3}{4}$) and its exact interval of convergence.

**Solution.** We apply the ratio test with $a_n = \frac{5^n}{n^2}(4x-3)^{2n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5^{n+1}}{(n+1)^2}(4x-3)^{2n+2}\frac{n^2}{5^n(4x-3)^{2n}}\right| = 5|4x-3|^2\frac{n^2}{(n+1)^2} \to 5|4x-3|^2$ as $n\to\infty$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{5^n}{n^2}(4x-3)^{2n}$ converges if $5|4x-3|^2 < 1$. To focus on $x$, we can rewrite this as $|4x-3| < \frac{1}{\sqrt{5}}$ or, equivalently, $\left|x-\frac{3}{4}\right| < \frac{1}{4\sqrt{5}}$.

In this latter form, we see that the radius of convergence is $\frac{1}{4\sqrt{5}}$.

The ratio test is inconclusive for $\left|x-\frac{3}{4}\right| = \frac{1}{4\sqrt{5}}$ or, equivalently, $x = \frac{3}{4} - \frac{1}{4\sqrt{5}}$ and $x = \frac{3}{4} + \frac{1}{4\sqrt{5}}$:

- $x = \frac{3}{4} + \frac{1}{4\sqrt{5}}$: $\sum_{n=1}^{\infty} \frac{5^n}{n^2}\left(\frac{1}{\sqrt{5}}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is the $p$-series with $p=2$ which converges (because $p>1$).
- $x = \frac{3}{4} - \frac{1}{4\sqrt{5}}$: $\sum_{n=1}^{\infty} \frac{5^n}{n^2}\left(-\frac{1}{\sqrt{5}}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is the same series and so converges as well.

Combined, the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{5^n}{n^2}(4x-3)^{2n}$ is $\left[\frac{3}{4} - \frac{1}{4\sqrt{5}}, \frac{3}{4} + \frac{1}{4\sqrt{5}}\right]$.