Notes for Lecture 1

Wed, 8/21/2024

Review: Our zoo of functions

- polynomials
  x^2, x^3, $7x^4 - x + 2$, ...
- rational functions
  $\frac{1}{x+1}$, $\frac{x^2 - 2x - 3}{x^3 + 7}$, ...
- power functions
  x^2, $x^{1/2} = \sqrt{x}$, $x^{-1/2} = \frac{1}{\sqrt{x}}$, ...
- exponentials
  2^x, e^x, ...
- logarithms
  $\ln(x) = \log_e(x)$, $\log_2(x)$, ...
- trigonometric functions
  $\sin(x)$, $\cos(x)$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, ...
- inverse trig functions
  $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, ...

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

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Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x) g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}.$$

Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x} (5x^3 + 7x^2 + 2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \sin(5x^3 + 7x^2 + 2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} (x^3 + 2x) \sin(5x^3 + 7x^2 + 2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x} (5x^3 + 7x^2 + 2) = 15x^2 + 14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \sin(5x^3 + 7x^2 + 2) = (15x^2 + 14x) \cos(5x^3 + 7x^2 + 2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} (x^3 + 2x) \sin(5x^3 + 7x^2 + 2)$
$= (3x^2 + 2) \sin(5x^3 + 7x^2 + 2) + (x^3 + 2x)(15x^2 + 14x) \cos(5x^3 + 7x^2 + 2)$

Example 3. Find $\frac{\mathrm{d}}{\mathrm{d}x} \tan(x)$ using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule.

Example 4. What is $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x)$?

Why? Can you explain why this is the case?
(One way to see this is to recall that $e^{\ln(x)} = x$ and to differentiate both sides. What do you conclude?)

Review: Integration and areas

If $f(x) \geqslant 0$ for $x \in [a, b]$, then

$$\int_a^b f(x) \mathrm{d}x$$

is defined as the area enclosed by the graph of $f(x)$ and the x-axis between $x = a$ and $x = b$.

Can you explain how the integral $\int_a^b f(x)\,\mathrm{d}x$ is constructed from sums $\sum f(x)\Delta x$?
(Here, we are summing over rectangles of width Δx between a and b; at position x their height is roughly $f(x)$.)

Comment. Σ is a capital sigma, and just means "sum". Don't worry about it for now. We will see it again later.

Review: Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the two operations

(a) differentiation and

(b) integration,

which, at first glance, look like they might be of a rather different nature. Roughly, it shows that these two operations are inverses of each other.

Theorem 5. (Fundamental Theorem of Calculus, part 2)

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

if $F(x)$ is an antiderivative of $f(x)$.

The "first part" of the Fundamental Theorem of Calculus is the statement that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

Modulo details (such as whether $f(x)$ is continuous or differentiable), can you conclude this "first part" from the "second part" above?
(Hint: in the "second part", replace b by x and differentiate both sides with respect to x.)

Example 6. Compute $\displaystyle\int_1^2 x\,\mathrm{d}x$ in two ways: first, making a sketch and using the definition as an area; then, using the Fundamental Theorem of Calculus.

Solution.

(a) Make a sketch! The area in question consists of a 1×1 square (area 1) with exactly half a 1×1 square (area $1/2$) on top. Hence, the total area is $1 + \frac{1}{2} = \frac{3}{2}$.

(b) $\displaystyle\int_1^2 x\,\mathrm{d}x = \left[\frac{1}{2}x^2\right]_1^2 = \frac{1}{2}\cdot 2^2 - \frac{1}{2}\cdot 1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

Example 7. $\displaystyle\int_0^\pi \sin(x)\,\mathrm{d}x =$

Why is $\displaystyle\int_0^{2\pi} \sin(x)\,\mathrm{d}x = 0$? Explain geometrically in terms of areas.

Review: Antiderivatives (or indefinite integrals)

Definition 8. If $f(x) = F'(x)$ then we say that $F(x)$ is an **antiderivative** of $f(x)$.

We write: $\int f(x)\mathrm{d}x = F(x) + C$

This is also called the **indefinite integral** of $f(x)$.

Comment. The notation using the integral sign makes sense because of the Fundamental Theorem of Calculus.

Example 9. $\int x\,\mathrm{d}x = \frac{1}{2}x^2 + C$

Example 10. $\int x^a\,\mathrm{d}x = \frac{1}{a+1}x^{a+1} + C$

Comment. Note that the case $a = -1$ is special. What happens in that case?

Example 11. $\int \frac{1-x}{x^3}\,\mathrm{d}x = \int (x^{-3} - x^{-2})\mathrm{d}x = -\frac{1}{2}x^{-2} + x^{-1} + C$

Example 12. $\int x\sqrt{x}\,\mathrm{d}x = \int x^{3/2}\mathrm{d}x = \frac{2}{5}x^{5/2} + C$

Substitution

The following is a first example for which the antiderivative is not so readily obtained by reversing basic rules of differentiation.

Example 13. Determine $\int x\sqrt{x^2+1}\mathrm{d}x$ by substituting $u = x^2 + 1$.

Solution. We need to substitute all occurences of x in the integral, including the $\mathrm{d}x$.

To substitute the latter, note that if $u = x^2 + 1$ then the derivative with respect to x is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x.$$

Solving for $\mathrm{d}x$, we find $\mathrm{d}x = \frac{1}{2x}\mathrm{d}u$.

Substituting in the integral, we therefore find

$$\int x\sqrt{x^2+1}\mathrm{d}x \;=\; \int x\sqrt{u}\,\frac{1}{2x}\mathrm{d}u = \frac{1}{2}\int \sqrt{u}\,\mathrm{d}u = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

Note how in the final step, we substituted back $u = x^2 + 1$ to get the desired antiderivative in terms of x.

Comment. We could have been slightly more efficient by directly substituting $x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.

Substitution, cont'd

In general, finding the right substitution can be tricky and there are no straightforward rules that work for every integral. However, for our integrals there is usually a natural choice.

On the other hand, there is often more than one way to substitute successfully (see next example).

Example 14. (cont'd) Determine $\int x\sqrt{x^2+1}\mathrm{d}x$

(a) by substituting $u = x^2+1$, and

(b) by substituting $u = \sqrt{x^2+1}$.

(c) Explain why the substitution $u = x^2+1$ is particularly natural.

Solution.

(a) Since $u = x^2+1$, we have $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ so that $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
The integral therefore becomes

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

(b) Since $u = \sqrt{x^2+1}$, we have $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x}{u}$ so that $x\mathrm{d}x = u\mathrm{d}u$.
The integral therefore becomes

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int u\cdot u\,\mathrm{d}u = \int u^2\,\mathrm{d}u = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

(c) The $\sqrt{x^2+1}$ in the integrand will become $\sqrt{u}$ (of course) while the remaining x in the integrand is (up to a factor of 2) exactly the derivative of x^2+1 (and so will disappear when we bring in $\mathrm{d}u$ to replace $\mathrm{d}x$). With a bit of practice, this allows us to immediately see that our substitution will be a success.

Substitution in definite integrals

If we want to apply substitution in a definite integral like $\int_a^b f(x)\mathrm{d}x$, we have two options:

(a) We can first compute the indefinite integral $\int f(x)\mathrm{d}x$ using substitution.
If the result is $F(x) + C$, then $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$.

(b) Or, we can substitute the limits a and b as well to get a new definite integral $\int_c^d g(u)\mathrm{d}u$.

Often you can choose either approach. But for certain problems it might not be possible to compute the indefinite integral explicitly (so the first approach won't work), yet it can be useful obtain a substituted new integral using the second approach.

Example 15. Determine $\int_0^1 x\sqrt{x^2+1}\mathrm{d}x$.

Solution. Let us illustrate both of the two approaches just mentioned:

(a) In the previous example, we already worked out that $\int x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{3}(x^2+1)^{3/2}+C$. It follows that

$$\int_0^1 x\sqrt{x^2+1}\mathrm{d}x = \left[\frac{1}{3}(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}2^{3/2}-\frac{1}{3}.$$

(b) As in the previous example, we will substitute $u=x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x}=2x$, we get $x\mathrm{d}x=\frac{1}{2}\mathrm{d}u$. Moreover, if $x=0$ (lower limit) then $u=x^2+1=0^2+1=1$. And if $x=1$ (upper limit) then $u=x^2+1=1^2+1=2$. The definite integral therefore becomes

$$\int_0^1 x\sqrt{x^2+1}\mathrm{d}x \;=\; \int_1^2 \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \left[\frac{1}{3}u^{3/2}\right]_1^2 = \frac{1}{3}2^{3/2}-\frac{1}{3}.$$

Comment. While it looks like the first approach was much quicker, that's only because we had already computed the indefinite integral in the previous example. In general, if we haven't already done so, the second approach is a bit more direct (but requires us to pay attention to the limits of the integral).

Review 16. $\int \frac{1}{x}\mathrm{d}x = \ln|x|+C$

To verify, use $\ln|x| = \begin{cases} \ln(x), & \text{if } x>0,\\ \ln(-x), & \text{if } x<0,\end{cases}$ to differentiate the right-hand side.

Example 17. Determine $\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t$.

Here, a very natural substitution is $u=2-\cos(t)$.

(Note how we can already anticipate that the derivative will nicely take care of the $\sin(t)$ in the integrand.)

Solution. We again have the choice of either substituting without limits or with limits:

(a) We substitute $u=2-\cos(t)$. Since $\frac{\mathrm{d}u}{\mathrm{d}t}=\sin(t)$, we get $\sin(t)\mathrm{d}t=\mathrm{d}u$. Hence,

$$\int \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int \frac{1}{u}\mathrm{d}u = \ln|u|+C = \ln|2-\cos(t)|+C.$$

Hence (by the Fundamental Theorem of Calculus)

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \Big[\ln|2-\cos(t)|\Big]_0^\pi = \ln(3).$$

(b) Alternatively, once we feel comfortable with integration, we can do this substitution in a single step by also adjusting the boundaries. When $t=0$ we have $u=2-\cos(0)=1$, and when $t=\pi$ we have $u=2-\cos(\pi)=3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{1}{u}\mathrm{d}u = \Big[\ln|u|\Big]_1^3 = \ln(3).$$

Extra practice

Example 18. Determine $\int x^3\sqrt{x^2+1}\mathrm{d}x$.

Solution. We substitute $u = x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we have $x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
Note that there will be x^2 left into the integral which we replace with $x^2 = u-1$.
The integral therefore becomes:

$$\begin{aligned}\int x^3\sqrt{x^2+1}\mathrm{d}x &= \int \frac{1}{2}x^2\sqrt{u}\,\mathrm{d}u = \int \frac{1}{2}(u-1)\sqrt{u}\,\mathrm{d} \\ &= \frac{1}{2}\int (u^{3/2}-u^{1/2})\,\mathrm{d}u = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C\end{aligned}$$

Comment. If you prefer, the final answer could be rewritten as $\frac{1}{15}(x^2+1)^{3/2}(3x^2-2)+C$.

Example 19. $\int \frac{\mathrm{d}x}{x\ln(x)} =$

First, use the substitution $u = \ln(x)$. (Can you already see why this is a good choice?)

Then, for practice, use $u = \frac{1}{\ln(x)}$ and see if you can get the same final answer.

(For more complicated integrals, finding the "best" substitution is quite an art form. For the integrals we are concerned with, there is always a natural choice.)

Example 20. $\int_2^4 \frac{\mathrm{d}x}{x\ln(x)} =$

To reduce work, use the previous problem and the fundamental theorem.

Example 21. $\int x\sin(x^2+3)\,\mathrm{d}x =$

Example 22. $\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)\mathrm{d}x =$

Example 23. $\int 3x^5\sqrt{x^3+1}\,\mathrm{d}x =$

First, use the substitution $u = x^3+1$ (that's the most natural choice).

Then, for practice, use $u = \sqrt{x^3+1}$ and see if you can get the same final answer.

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Review. There will be a first quiz in lab on Thursday. One of the two problems will be to compute an integral like the following by substitution:

$$\int \cos^5(3t)\sin(3t)\mathrm{d}t$$

Example 24.

(a) Determine $\displaystyle\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t$.

(b) Determine $\displaystyle\int_0^\pi \frac{\sin^3(t)}{2-\cos(t)}\mathrm{d}t$.

Solution.

(a) **(again)** We substitute $u=2-\cos(t)$. When $t=0$ we have $u=2-\cos(0)=1$, and when $t=\pi$ we have $u=2-\cos(\pi)=3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t=\int_1^3 \frac{1}{u}\mathrm{d}u=\Big[\ln|u|\Big]_1^3=\ln(3).$$

(b) We substitute $u=2-\cos(t)$ again. When $t=0$ we have $u=2-\cos(0)=1$, and when $t=\pi$ we have $u=2-\cos(\pi)=3$. This time, there will be $\sin^2(t)$ left over in the integral so we need to rewrite this in terms of u.

Note that $\cos(t)=2-u$. (At this point, we could solve for t to get $t=\arccos(2-u)$ and use this to substitute away any remaining t. However, we would get $\sin^2(t)=\sin^2(\arccos(2-u))$ which is not pleasant and would have to be simplified. We get this simplification for free by proceeding slightly differently.) Recall that $\cos^2(t)+\sin^2(t)=1$ so that $\sin^2(t)=1-\cos^2(t)=1-(2-u)^2=-u^2+4u-3$.

Therefore,

$$\int_0^\pi \frac{\sin^3(t)}{2-\cos(t)}\mathrm{d}t=\int_1^3 \frac{\sin^2(t)}{u}\mathrm{d}u=\int_1^3 \frac{-u^2+4u-3}{u}\mathrm{d}u=\int_1^3\left(-u+4-\frac{3}{u}\right)\mathrm{d}u=\ldots=4-3\ln(3).$$

Areas enclosed by curves

Theorem 25. The area enclosed by the curves $y=f(x)$ and $y=g(x)$, between $x=a$ and $x=b$, is given by

$$\int_a^b [f(x)-g(x)]\,\mathrm{d}x$$

provided that $f(x)\geqslant g(x)$ (for all $x\in[a,b]$).

[Note that the area is always $\int_a^b|f(x)-g(x)|\,\mathrm{d}x$ but to work with the absolute value, we need to break the problem into subcases according to whether $f(x)-g(x)\geqslant 0$ or $f(x)-g(x)\leqslant 0$.]

Example 26. What is the area enclosed by the curves $y=\cos(x)$, $y=1$, $x=0$, $x=2\pi$?

First, write down an integral and compute its value, then look at your sketch (always make a quick sketch!) and confirm that your answer makes rough sense.

Solution. Note that $y=1$ describes a horizontal line while $x=0$ and $x=2\pi$ are two vertical lines. For our area, $y=1$ lies above $y=\cos(x)$. Therefore, the area in question is

$$\int_0^{2\pi}(1-\cos(x))\mathrm{d}x=\Big[x-\sin(x)\Big]_0^{2\pi}=(2\pi-\sin(2\pi))-(0-\sin(0))=2\pi.$$

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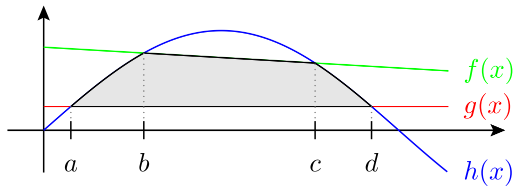
Example 27. Consider the plot below. What is the area enclosed by the curves $y = f(x)$, $y = g(x)$ and $y = h(x)$?



Solution. We split up the area into three smaller regions in which we can apply Theorem 25. The area is

$$\int_a^b [h(x) - g(x)]\mathrm{d}x + \int_b^c [f(x) - g(x)]\mathrm{d}x + \int_c^d [h(x) - g(x)]\mathrm{d}x.$$

Comment. In this case, we can alternatively (and equivalently) write the area as the difference

$$\int_a^d [h(x) - g(x)]\mathrm{d}x - \int_b^c [h(x) - f(x)]\mathrm{d}x.$$

Example 28. What is the area enclosed by the curves $y = 2 - x^2$, $y = -x$?

(a) First, make a sketch!

(b) Find intersections of the curves.

(c) Write down the integral for the area of interest.

(d) Evaluate the integral.

Solution.

(a) Do it! This should always be our first step.

(b) Since the equations for both curves are of the form $y = \ldots$, we can find the (x coordinates of the) intersections by setting the right-hand sides of the equations equal:

$$2 - x^2 = -x \quad \Longrightarrow \quad x^2 - x - 2 = 0 \quad \Longrightarrow \quad x = -1, 2$$

For the final step, we solved the quadratic equation (for instance, using the quadratic formula).

[It's not needed for the remaining parts, but we can get the corresponding y-coordinates from either $y = 2 - x^2$ or $y = -x$. The latter is simpler and we find that the two intersections are $(-1, 1)$ and $(2, -2)$.]

(c) This tells us (look at sketch!) that our area extends from $x = -1$ to $x = 2$ and that the curve $y = 2 - x^2$ is the upper boundary while $y = -x$ is the lower boundary. Therefore, the area is

$$\int_{-1}^2 ((2 - x^2) - (-x))\mathrm{d}x.$$

(d) We compute the area as

$$\int_{-1}^2 ((2 - x^2) - (-x))\mathrm{d}x = \int_{-1}^2 (2 + x - x^2)\mathrm{d}x = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^2 = \ldots = \frac{9}{2}.$$

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Example 29. What is the area enclosed by the curves $y = 3 - x^2$, $y = 2$, $y = -1$?

First, make a sketch like we did in class!

For this problem, we then have a choice of whether we cut up our area into tiny vertical rectangles (with width $\mathrm{d}x$) or into tiny horizontal rectangles (with width $\mathrm{d}y$). Below, we do both. Of course, the final answer will be the same.

Solution. (vertical slicing) We first need to find the intersections of the parabola $y = 3 - x^2$ with each of $y = 2$ and $y = -1$.

- $y = 3 - x^2$ and $y = 2$: solving $3 - x^2 = 2$, we get $x^2 = 1$ and so $x = \pm 1$.
- $y = 3 - x^2$ and $y = -1$: solving $3 - x^2 = -1$, we get $x^2 = 4$ and so $x = \pm 2$.

The area is therefore given by the following three integrals:

$$\int_{-2}^{-1} ((3 - x^2) - (-1))\mathrm{d}x + \int_{-1}^{1} (2 - (-1))\mathrm{d}x + \int_{1}^{2} ((3 - x^2) - (-1))\mathrm{d}x$$

You might notice that the first and last integral must be equal, while the second integral is just computing the area of a rectangle of height $2 - (-1) = 3$ and width $1 - (-1) = 2$ (and so its area is $3 \cdot 2 = 6$).

Whether or not you use any simplifications, the above integrals evaluate to $\frac{5}{3} + 6 + \frac{5}{3} = \frac{28}{3}$ (do it!).

Solution. (horizontal slicing) This time we don't need to compute the intersections because the area (see sketch!) clearly extends from $y = -1$ to $y = 2$.

Since we are working in y-direction, we rewrite $y = 3 - x^2$ as $x^2 = 3 - y$ or $x = \pm\sqrt{3 - y}$ (note that $x = \sqrt{3 - y}$ describes the right half, while $x = -\sqrt{3 - y}$ describes the left half).

So the horizontal slice has length $\sqrt{3 - y} - (-\sqrt{3 - y}) = 2\sqrt{3 - y}$. Accordingly, the total area is

$$\int_{-1}^{2} (\sqrt{3 - y} - (-\sqrt{3 - y}))\mathrm{d}y = 2\int_{-1}^{2} \sqrt{3 - y}\,\mathrm{d}y = 2\left[-\frac{2}{3}(3 - y)^{3/2}\right]_{-1}^{2} = -\frac{4}{3}(1 - 4^{3/2}) = -\frac{4}{3}(1 - 8) = \frac{28}{3}.$$

Volumes using cross-sections

We have computed areas of certain regions by cutting them into tiny (typically vertical) pieces of width $\mathrm{d}x$ and some height $h(x)$. If the region extends from $x=a$ to $x=b$, then the total area is the sum of the areas of these pieces (each has area $h(x)\mathrm{d}x$) and therefore given by

$$\text{area} = \int_a^b h(x)\mathrm{d}x.$$

We now use the same idea to compute volumes:

Please have a look at the Section 6.1 in the book for all the pretty pictures and detailed explanations. Below is just a summary which probably doesn't make too much sense unless you have been in class or have read through the beginning of Section 6.1.

- The volume of a cylindrical solid is its base area times its height.
- The idea for computing the volume of more general solids is to cut it into little slices that are approximately cylindrical.
- Suppose that the slice at position x has cross-sectional area $A(x)$. If we are slicing with a width of $\mathrm{d}x$, then this slice has roughly volume $A(x)\mathrm{d}x$.
- Summing the volumes of all these slices leads to the follwing formula of the volume of the general solid:

$$\text{vol} = \int_a^b A(x)\mathrm{d}x$$

Note: It is usually up to us to introduce coordinates for the position x. The formula above assumes that our solid extends from $x=a$ to $x=b$ in these coordinates.

Example 30. Derive the formula for the volume of a pyramid of height h whose base is a square with sides of length a.

You might remember that the volume is $\frac{1}{3}a^2h$ but the point of this example is that we can actually find this formula without knowing it by slicing the pyramid (the easiest way to slice is horizontally, so that each cross-section is again just a square).

Solution. We cut the pyramid into cross-sections parallel to its base. Let x, between 0 and h, denote how far we have gone from the tip $(x=0)$ down to the base of the pyramid $(x=h)$. Then, at x, the cross-section is a square with side length $a\frac{x}{h}$ (make a sketch!). This cross-section has area $A(x) = (\frac{a}{h}x)^2$. Therefore, the volume is

$$\text{vol} = \int_0^h A(x)\mathrm{d}x = \int_0^h \left(\frac{a}{h}x\right)^2 \mathrm{d}x = \frac{a^2}{h^2}\int_0^h x^2\mathrm{d}x = \frac{a^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h = \frac{a^2}{h^2}\cdot\left(\frac{1}{3}h^3 - 0\right) = \frac{1}{3}a^2h.$$

Comment. Introducing the extra variable x can be done in different ways and it requires some practice to make the easiest choice. For instance, if we had let x, again between 0 and h, denote how far we have gone from the base $(x=0)$ to the tip of the pyramid $(x=h)$, then, at x, the cross-section is a square with side length $a\frac{x-h}{h}$. We would get the same in the end but with a little bit more algebra.

Solids of revolution

Consider a region (for instance, the region enclosed by a bunch of curves). A solid of revolution is what we obtain when revolving this region about a given line.

Again, Section 6.1 in the book contains lots of helpful illustrations.

- Suppose the region is the area between a curve $R(x)$ and the x-axis, between $x=a$ and $x=b$.
- Further, suppose that we revolve this region about the x-axis. Then, slicing vertically, the cross-sections are disks (circles with the interior) with radius $R(x)$ (so that the cross-sectional area is $A(x)=\pi R(x)^2$).
- Hence, the volume of the resulting solid is

$$\text{vol} = \int_a^b A(x)\mathrm{d}x = \int_a^b \pi R(x)^2\,\mathrm{d}x.$$

Example 31. Consider the region enclosed by the curves $y=\sqrt{x}$, $y=0$, $x=1$, $x=4$. If we revolve this region about the x-axis, what is the volume of the resulting solid?

Solution. Make a sketch! The solid should look roughly like a solid coffee mug (no room for coffee) without handles.

The cross-section at x is a disk with radius $R(x)=\sqrt{x}$ and so has area $\pi R(x)^2=\pi x$. If we give this disk a thickness of $\mathrm{d}x$, then its volume is $\pi R(x)^2\,\mathrm{d}x=\pi x\mathrm{d}x$. The total volume is

$$\text{vol} = \int_1^4 \pi R(x)^2\,\mathrm{d}x = \int_1^4 \pi x\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2\right]_1^4 = \pi\left(8-\frac{1}{2}\right) = \frac{15}{2}\pi.$$

Example 32. Consider the region enclosed by the curves $y=\sqrt{x}$, $y=1$, $x=1$, $x=4$. If we revolve this region about the line $y=1$, what is the volume of the resulting solid?

Solution. Again, make a sketch! This solid should look roughly like a bullet.

The cross-section at x now is a disk with radius $R(x)=\sqrt{x}-1$ and so has area $\pi R(x)^2=\pi(\sqrt{x}-1)^2$. If we give this disk a thickness of $\mathrm{d}x$, then its volume is $\pi R(x)^2\,\mathrm{d}x=\pi x\mathrm{d}x$. The total volume is

$$\text{vol} = \int_1^4 \pi R(x)^2\mathrm{d}x = \int_1^4 \pi(\sqrt{x}-1)^2\mathrm{d}x = \int_1^4 \pi(x-2\sqrt{x}+1)\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2-\frac{4}{3}x^{3/2}+x\right]_1^4 = \pi\left(\frac{4}{3}-\frac{1}{6}\right) = \frac{7}{6}\pi.$$

Example 33. Consider (again) the region enclosed by the curves $y=\sqrt{x}$, $y=1$, $x=1$, $x=4$. If we revolve this region about the x-axis, what is the volume of the resulting solid?

The solid should look roughly like the coffee mug from Example 31 with a cylindrical hole drilled out.

What do the cross-sections look like now?

Your final answer should be $\frac{9}{2}\pi$.

Solution. The solid should look roughly like the coffee mug from Example 31 with a cylindrical hole drilled out.

The cross-section at x now is a **washer** with outer radius $R(x)=\sqrt{x}$ and inner radius $r(x)=1$ so that its area is $\pi R(x)^2-\pi r(x)^2=\pi x-\pi=\pi(x-1)$. If we give this washer a thickness of $\mathrm{d}x$, then its volume is $\pi(x-1)\mathrm{d}x$. The total volume is

$$\text{vol} = \int_1^4 \pi(x-1)\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2-x\right]_1^4 = \pi\left(4-\left(-\frac{1}{2}\right)\right) = \frac{9}{2}\pi.$$

Comment. In this simple case, we could also obtain the final answer from Example 31 by simply subtracting the volume of the cylinder (base area $\pi\cdot 1^2$ times height $4-1=3$ gives a volume of 3π) that is "drilled out".

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Volumes using cylindrical shells

See Section 6.2 in our book for nice illustrations of cylindrical shells!

Example 34. Consider the region bounded by the curves $y=x^2$, $y=0$ and $x=2$. Determine the volume of the solid generated by revolving this region about the y-axis

(a) using horizontal cross-sections of the region (turning into washers), and

(b) using vertical cross-sections of the region (turning into **cylindrical shells**).

Solution. First, make a sketch as we did in class!

(a) Our region extends from $y=0$ to $y=4$. The cross-section at y (a line which extends from $x=\sqrt{y}$ to $x=2$) turns into a washer with outer radius $R(y)=2$ and inner radius $r(y)=\sqrt{y}$, which has area $\pi R(y)^2 - \pi r(y)^2 = \pi(4-y)$. If we give this disk a thickness of $\mathrm{d}y$, then its volume is $\pi(4-y)\mathrm{d}y$. The total volume is

$$\mathrm{vol} = \int_0^4 \pi(4-y)\,\mathrm{d}y = \pi\left[4y - \frac{1}{2}y^2\right]_0^4 = 8\pi.$$

(b) Our region extends from $x=0$ to $x=2$. The cross-section at x (a line which extends from $y=0$ to $y=x^2$) turns into a cylindrical shell with radius $r(x)=x$ and height $h(x)=x^2$ and so has area $2\pi r(x)\cdot h(x) = 2\pi x^3$. If we give this shell a thickness of $\mathrm{d}x$, then its volume is $2\pi x^3\mathrm{d}x$. The total volume is

$$\mathrm{vol} = \int_0^2 2\pi x^3\,\mathrm{d}x = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi.$$

The following is a similar but slightly beefed up example.

Example 35. Consider the region bounded by the curves $y=x^2$, $y=1$ and $x=2$. Determine the volume of the solid generated by revolving this region about the line $x=-1$.

(a) using horizontal cross-sections of the region (turning into disks), and

(b) using vertical cross-sections of the region (turning into cylindrical shells).

Solution. As always, start with a sketch! Note that $y=1$ is now describing the bottom: the bottom-left corner of the region is $(1,1)$, the bottom-right corner is $(2,1)$, and the top-right corner is $(2,4)$.

(a) Our region extends from $y=1$ to $y=4$. The cross-section at y (a line which extends from $x=\sqrt{y}$ to $x=2$) turns into a washer with outer radius $R(y)=2-(-1)=3$ and inner radius $r(y)=\sqrt{y}-(-1)=\sqrt{y}+1$, which has area $\pi R(y)^2 - \pi r(y)^2 = 9\pi - \pi(\sqrt{y}+1)^2 = \pi(8-2\sqrt{y}-y)$. If we give this disk a thickness of $\mathrm{d}y$, then its volume is $\pi(8-2\sqrt{y}-y)\mathrm{d}y$. The total volume is

$$\mathrm{vol} = \int_1^4 \pi(8-2\sqrt{y}-y)\,\mathrm{d}y = \pi\left[8y - \frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_1^4 = \pi\left(\frac{40}{3} - \frac{37}{6}\right) = \frac{43}{6}\pi.$$

(b) Our region extends from $x=1$ to $x=2$. The cross-section at x (a line which extends from $y=1$ to $y=x^2$) turns into a cylindrical shell with radius $r(x)=x-(-1)=x+1$ and height $h(x)=x^2-1$ and so has area $2\pi r(x)\cdot h(x) = 2\pi(x+1)(x^2-1)$. If we give this shell a thickness of $\mathrm{d}x$, then its volume is $2\pi(x+1)(x^2-1)\mathrm{d}x$. The total volume is

$$\mathrm{vol} = \int_1^2 2\pi(x+1)(x^2-1)\,\mathrm{d}x = \int_1^2 2\pi(x^3+x^2-x-1)\,\mathrm{d}x = 2\pi\left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 - x\right]_1^2 = \frac{43}{6}\pi.$$

Example 36. (extra) Consider the region enclosed by the curves

$$y = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 4.$$

Determine the volume of the solid generated by revolving this region about the x-axis

(a) using vertical cross-sections of the region, and

(b) using horizontal cross-sections of the region.

Solution. As always, start with a sketch!

(a) The volume is

$$\int_0^4 \pi(\sqrt{x})^2 \, \mathrm{d}x = \pi \int_0^4 x \, \mathrm{d}x = \pi \left[\frac{1}{2}x^2\right]_0^4 = 8\pi.$$

(b) We operate between $y = 0$ and $y = \sqrt{4} = 2$. A slice at height y has width $4 - y^2$ (note that $y = \sqrt{x}$ implies $x = y^2$). Giving this slice a thickness of $\mathrm{d}y$ and revolving about the x-axis, we obtain a cylindrical shell with approximate volume

$$(\text{circumference}) \times (\text{width}) \times (\text{thickness}) = (2\pi y)(4 - y^2) \, \mathrm{d}y.$$

"Summing" all these volumes, we get

$$\int_0^2 2\pi y(4 - y^2) \, \mathrm{d}y = 2\pi \left[2y^2 - \frac{1}{4}y^4\right]_0^2 = 2\pi(8 - 4) = 8\pi.$$

Sure enough, the volume is exactly what we calculated before.

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Review. Which equation describes the circle of radius r centered at the origin?

Solution. The circle consists of all points (x, y) that satisfy $x^2 + y^2 = r^2$.
This is just the Pythagorean theorem (make a sketch to make sure this is clear to you).

Example 37. We wish to compute the volume of a ball of radius r.

(a) Which region can we revolve to obtain a ball as our solid of revolution?

(b) Setup the appropriate integral for the volume and evaluate it.

For practice, you can compute it using disks/washers as well as using cylindrical shells.

Solution.

(a) We can revolve a half-circle to end up with a ball. A convenient choice is to take the region between $y = \sqrt{r^2 - x^2}$ and revolve it about the x-axis. This is what we will use for the next part.

(b) Taking vertical cross-sections, we get disks after revolving and the total volume is

$$\int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 \mathrm{d}x = \int_{-r}^{r} \pi(r^2 - x^2)\, \mathrm{d}x = \pi\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} = \frac{4}{3}\pi r^3$$

Sure enough, this is the formula for the volume of a ball that we have seen before (though our memory might be foggy on the exact formula).

Alternatively. It is more complicated here but, for practice, we can also take horizontal cross-sections in which case we get cylindrical shells after revolving. Lets take the cross-section at height y where y is between 0 and r. This cross-section extends from $x = -\sqrt{r^2 - y^2}$ to $x = \sqrt{r^2 - y^2}$ and therefore has width $2\sqrt{r^2 - y^2}$. Giving this slice a thickness of $\mathrm{d}y$ and revolving about the x-axis, we obtain a cylindrical shell with approximate volume

$$(\text{circumference}) \times (\text{width}) \times (\text{thickness}) = (2\pi y)\left(2\sqrt{r^2 - y^2}\right) \mathrm{d}y.$$

"Summing" all these volumes, we get

$$\int_0^r (2\pi y)\left(2\sqrt{r^2 - y^2}\right) \mathrm{d}y = 4\pi \int_0^r y\sqrt{r^2 - y^2}\, \mathrm{d}y = 4\pi\left[-\frac{1}{3}(r^2 - y^2)^{3/2}\right]_0^r = \frac{4}{3}\pi r^3.$$

Here, we substituted $u = r^2 - y^2$ to compute the integral (do it!). The final volume is, of course, the same we calculated before.

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Arc length

Example 38. What is the length of the curve $y=2x$, for $0\leqslant x\leqslant 4$?

Make a sketch and use Pythagoras.

Solution. The curve is the hypothenuse of a right triangle with shorter sides of length 4 (in x-direction) and 8 (in y-direction). Therefore its length is $\sqrt{4^2+8^2}=\sqrt{80}=4\sqrt{5}$.

(arc length) The length of a general curve $y=f(x)$, for $a\leqslant x\leqslant b$, is given by

$$\int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

Why? To see how we can arrive at this formula, we proceeded as follows:

- We chop the x-axis into little pieces of width $\mathrm{d}x$ and look at the corresponding pieces of our graph.
- Suppose we are looking at our graph near x. If we zoom in plenty, then the tiny portion of the graph we see begins to look roughly like a line with slope $f'(x)$.
- We can compute the length of a segment of this line as we did in Example 38 by using Pythagoras. If the segment extends $\mathrm{d}x$ horizontally, then it extends $f'(x)\mathrm{d}x$ vertically (make a sketch!). [We can also write $f'(x)\mathrm{d}x=\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x=\mathrm{d}y$.]

  By Pythagoras, our piece of the line has length

  $$\sqrt{(\mathrm{d}x)^2+(f'(x)\,\mathrm{d}x)^2}=\sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$
- "Adding" all these little pieces, we obtain the formula above for the total length of the curve.

Example 39. **(again)** Using the integral formula, compute the length of the curve $y=2x$, for $0\leqslant x\leqslant 4$, again. Of course, the answer agrees with Example 38.

Solution. Here, $f(x)=2x$ so that $f'(x)=2$. Hence, the length is

$$\int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x=\int_0^4 \sqrt{1+2^2}\,\mathrm{d}x=4\sqrt{5}\approx 8.944.$$

Example 40. Compute the length of the curve $y=x^{3/2}$, for $0\leqslant x\leqslant 4$.

Before you compute the answer, make a sketch. Which curve should be longer: $y=x^{3/2}$ or $y=2x$?

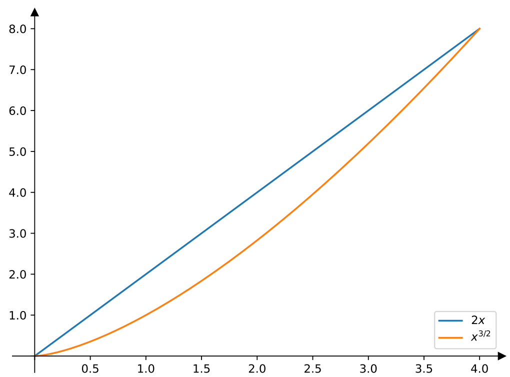
Solution. With $f(x)=x^{3/2}$, we have $f'(x)=\frac{3}{2}\sqrt{x}$. The length is

$$\int_0^4 \sqrt{1+\left(\frac{3}{2}\sqrt{x}\right)^2}\,\mathrm{d}x=\int_0^4 \sqrt{1+\frac{9}{4}x}\,\mathrm{d}x=\int_1^{10}\frac{4}{9}\sqrt{u}\,\mathrm{d}u=\frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}=\frac{8}{27}(\sqrt{1000}-1)\approx 9.073.$$

Here, we substituted $u=1+\frac{9}{4}x$. Make sure that this substitution is clear to you (including the change of boundaries: if $x=4$ then $u=?$).

Comparing the lengths. As for comparing the lengths, note that, on the interval $0 \leqslant x \leqslant 4$, both $y = x^{3/2}$ and $y = 2x$ begin at the point $(0, 0)$ and end at the point $(4, 8)$. Since a line is the shortest connection between two points, the arc length for $y = x^{3/2}$ had to be larger.

However, you can see that the difference is not much. This is confirmed by the plot below:



Example 41. Setup an integral for the circumference of a circle of radius r.

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Review. The (arc) length of a general curve $y = f(x)$, for $a \leqslant x \leqslant b$, is given by

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x.$$

Example 42. (extra) Using our new technology, compute the circumference of a circle of radius r.

Solution. Of course, the final answer has to be $2\pi r$.

- First, in order to work with a function, we consider the (upper) half circle.

  That half-circle is described by $y = \sqrt{r^2 - x^2}$ and the length of that curve (from $x = -r$ to $x = r$) is half of the circumference of our circle.

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-x}{\sqrt{r^2 - x^2}}$

- Hence, the circumference of our circle is given by the integral

$$2\int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} \mathrm{d}x = 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \mathrm{d}x = 2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}} \mathrm{d}x.$$

- Now, we substitute $u = x/r$.

  Why? Just looking at the integral, the reason for choosing this substitution might not be obvious. However, thinking about our actual problem, this substitution is very natural: it scales things by $1/r$ so that our circle gets rescaled to a circle of radius 1.

It is very common in applications that we need to change scales or coordinate systems. When dealing with integrals, we then need to perform the corresponding substitution.

Such changes of scale or coordinate systems for practical reasons is a second important reason why we need to be able to substitute. (So far we substituted as a means to mathematically simplify an integral.)

- To substitute, we compute $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{r}$, and so $\mathrm{d}x = r\,\mathrm{d}u$ [r is just a number—we can treat it like we would treat 7.]

  For the boundaries of the integral: if $x = -r$, then $u = -1$. If $x = r$, then $u = 1$.

  Since $x = ru$, we therefore get

$$2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}} \mathrm{d}x = 2\int_{-1}^{1} \sqrt{\frac{r^2}{r^2 - (ru)^2}} \cdot r\,\mathrm{d}u = 2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}} \mathrm{d}u.$$

- We can actually evaluate this final integral (more on such integrals later) if we recall that the derivative of $\arcsin(u)$ is $1/\sqrt{1 - u^2}$. Hence,

$$2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}} \mathrm{d}u = 2r\Big[\arcsin(u)\Big]_{-1}^{1} = 2r[\arcsin(1) - \arcsin(-1)] = 2\pi r.$$

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Physical work

Work is **force** times **distance**: $W = Fd$.

- F could be measured in lb and d in ft. Then W is conveniently measured in ft-lb.
- The SI units for F are N (newton), for d they are m (meter), and W is measured in Nm (newtonmeter) or joule (1 Nm = 1 joule ≈ 0.738 ft-lb).

Think about the work necessary to lift an object to a certain height.

Calculus comes in when the force F is not constant!

Example 43. Suppose we wish to lift a 100 lb piano from the ground to the top of a 20 ft building (for instance, by standing on the roof and pulling it up using a rope). The work required for that is

$$\text{work} = (100 \text{ lb})(20 \text{ ft}) = 2000 \,\text{ft-lb}.$$

This was easy because the force was constant througout the problem (the piano always weighed 100 lb). It is when the force varies (as in the next example) that we need our calculus skills and mastery of integrals.

Example 44. As before, we wish to lift a 100 lb piano from the ground to the top of a 20 ft building. We are doing so by standing on the roof and pulling it up using a rope. However, this time, we are using a rather heavy rope weighing 0.1 lb/ft and want to take that into account (just pulling up the rope, dangling to the ground, would require some work).

Think about the moment when the piano is x ft off the ground (and we want to pull it up by $\mathrm{d}x$ ft):

- We still need to pull up $20 - x$ ft.

  So, at that moment, the weight (piano plus rope) to be pulled up is $100 + 0.1(20 - x)$ lb.
- Hence, to pull up the piano by a tiny amount of $\mathrm{d}x$ feet, the amount of work needed is (roughly) $[100 + 0.1(20 - x)]\mathrm{d}x$ pound.

  [Assuming that $\mathrm{d}x$ is very small, the change in weight is insignificant, so that we can use $W = Fd$.]

To get the total amount of work (in ft-lb), we need to "add" up these small contributions from $x = 0$ to $x = 20$:

$$\text{work} = \int_0^{20} [100 + 0.1(20 - x)]\mathrm{d}x.$$

It only remains to calculate this integral (which is very simple in this case):

$$\text{work} = \int_0^{20} [102 - 0.1x]\mathrm{d}x = \left[102x - \frac{0.1}{2}x^2\right]_0^{20} = 2020 \text{ ft-lb}.$$

Comment. In this simple example, you can get away with not computing any integrals by arguing as follows: the rope weighs a total of $20 \cdot 0.1 = 2$ lb. On average, we need to lift it 10 ft so that the total amount of work needed to pull up the rope is 2 lb · 10 ft = 20 ft-lb. Added to the work required for just the piano (2000 ft-lb, see previous example), we get the total that we just computed.

Example 45. A conical container of radius 10 ft and height 30 ft is completely filled with water (the tip of the cone is at the bottom). How much work will it take to pump the water to a level of 2 ft above the cone's rim? [Water weighs $62.4\text{ lb}/\text{ft}^3$.]

Solution. Make a sketch! Let us denote with y (in ft) the vertical position in such a way that the tip is at $y=0$ and the rim is at $y=30$. We need to pump the water to the level $y=30+2=32$.

We consider a horizontal slice at height y and thickness dy:

- This slice needs to be lifted up $(30-y)+2=32-y$ (ft).
  (If we considered vertical slices, we could not make such a statement and therefore would be stuck here.)
- The slice is (almost) a disk with radius $r=\frac{10}{30}y=\frac{y}{3}$. Hence, its volume is $\pi r^2\,dy=\frac{\pi}{9}y^2dy$ (ft^3).
- The weight of the slice is $62.4\frac{\pi}{9}y^2\,dy$ (lb) and it needs to be lifted up $32-y$ (ft). That takes work of $(32-y)\cdot 62.4\frac{\pi}{9}y^2dy$ (ft-lb).
- "Adding" up, the total work required is

$$\int_0^{30}(32-y)\cdot 62.4\frac{\pi}{9}y^2dy=\ldots\approx 1.86\cdot 10^6\text{ ft-lb}.$$

Comment. Note that we rounded the answer to 3 significant digits, because we cannot expect more precision given that the weight of water per ft^3 is only given to us to 3 digits.

Example 46. Repeat Example 45 if...

(a) ...the container is not completely filled with water but only to a height of 15 ft.

(b) ...we stop pumping once the container is filled with water to a height of 15 ft.

Can you predict in which case the work required is larger?

Solution.

(a) In that case, the total work required is

$$\int_0^{15}(32-y)\cdot 62.4\frac{\pi}{9}y^2dy=\ldots\approx 508{,}000\text{ ft-lb}.$$

(b) In that case, the total work required is

$$\int_{15}^{30}(32-y)\cdot 62.4\frac{\pi}{9}y^2dy=\ldots\approx 1.35\cdot 10^6\text{ ft-lb}.$$

Comment. Note that the sum of these two is again $1.86\cdot 10^6$ ft-lb. Think about why that makes perfect sense!

Example 47. We want to "lift" a 1600 kg satellite from the ground into orbit, $20{,}000 \text{ km}$ above the surface. Let us compute the theoretic amount of work required to do so.

Context. These are actually typical values for a GPS satellite. For comparison, the ISS has an average altitude about 400 km, while the moon is about $384{,}000 \text{ km}$ away. Light travels at the speed of about $300{,}000 \text{ km/s}$ (sound only at about 340 m/s).

Solution. First, let us gather the necessary background information:

- Initially, the satellite is sitting on the surface, about $d_1 = 6371 \text{ km}$ from the center of earth (for gravitation, earth behaves like all its mass is concentrated at its center).
  The goal is to bring the satellite to a distance $d_2 = 26{,}371 \text{ km}$ from the center of earth.
- The mass of the earth is about $m_E = 5.972 \cdot 10^{24} \text{ kg}$. The mass of our satellite is $m_S = 1600 \text{ kg}$.
- The physical law of attraction is $F = G\frac{m_E m_S}{d^2}$.
  It tells us the force of attraction between two masses (here, the satellite and earth) that are at distance d.
  Here, $G = 6.674 \cdot 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$ is the gravitational constant.
  Comment. Note that this force is not constant in our problem: the values of d range from $d = d_1$ to $d = d_2$.

Make a sketch! Similar to our approach Example 44, we now think about the moment when the satellite is at distance x from the center of the earth and about the amount of work needed to lift it by $\mathrm{d}x$.

- At that moment, the gravitational force is $G\frac{m_S m_E}{x^2}$.
- To lift up the satellite by a tiny amount of $\mathrm{d}x$, the amount of work needed is (roughly) $G\frac{m_S m_E}{x^2}\mathrm{d}x$ (force times distance).

Hence, the total amount of work is

$$\text{work} = \int_{d_1}^{d_2} G\frac{m_S m_E}{x^2}\mathrm{d}x = G m_S m_E \int_{d_1}^{d_2}\frac{\mathrm{d}x}{x^2} = G m_S m_E\left[-\frac{1}{x}\right]_{d_1}^{d_2} = G m_S m_E\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

Plugging in our values for G, m_S, m_E, d_1, d_2, we find

$$\text{work} = (6.674 \cdot 10^{-11}) \cdot (1600) \cdot (5.972 \cdot 10^{24})\left(\frac{1}{6371000} - \frac{1}{26371000}\right) \approx 7.59 \cdot 10^{10} \text{ joule}.$$

Comment. You could also let x be the distance from the surface. That works as well if you adjust things accordingly. (Do that as an exercise!)
Comment. If you choose $d = 6371 \text{ km}$ in the law of attraction, set one mass to be the mass $m_E = 5.972 \cdot 10^{24}$ kg of earth and the other to be 1 kg, then the resulting force is $F = 6.674 \cdot 10^{-11} \cdot \frac{5.972 \cdot 10^{24}}{(6.371 \cdot 10^6)^2} \approx 9.820 \text{ N}$, which you have surely met in other class before. Note that this force is an approximation and depends on the exact elevation (earth is not perfectly round). Usually, the value 9.81 N is used.

Example 48.

- What happens when we take the **limit** $d_2 \to \infty$ in the previous example? What does that mean physically?
- How does the previous problem change if the physical law of attraction was $F = G\frac{m_S m_E}{d}$? What happens now when we take the limit $d_2 \to \infty$?

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Example 49. (extra) The Pyramid of Cheops, built about 2560 BC, has been the tallest man-made structure in the world for over 3800 years. The pyramid was built to a height of 146m. Its base is a square with each side 230m in length.

(a) The pyramid is made out of limestone (1 cubicmeter of limestone has a mass of 2.3 tonnes). Assuming that the pyramid is solid, compute its mass (in tonnes) by using an integral.

(b) What was the mass of the pyramid when it was built to half its final height?

(c) The limestone blocks used usually have a mass of about 2.5 tonnes each. (Roughly) how many limestone blocks does the pyramid consist of?

(d) Compute the (theoretical) total amount of work (in joule) that was required in lifting all the blocks from the ground to their final position.

(e) Of course, the actual amount of work required was much higher; assume it was 50 times as high as the theoretical amount you just calculated. Further, assume that an Egyptian worker could perform work of about 2000 kilojoules per day. Based on these numbers and no holidays, how many workers would have been needed to construct the Pyramid of Cheops in 20 years?
Comment. That's just for the building part! Sourcing and transport of the blocks and all other stuff not included ... For perspective, assume a person consumes 2000 calories a day. That actually means $2000 \text{ kcal} \approx 8400 \text{ kJ}$, and that is an upper limit on how much work they can perform on a daily basis.

Solution.

(a) Let us denote with x (in m) the vertical position in such a way that the tip is at $x=0$ and the base is at $x=146$. That way, the cross-section at x has a width of $\frac{230}{146}x$ (so that the width is 0 when $x=0$, and the width is 230 when $x=146$). The volume of the pyramid is

$$\int_0^{146}\left(\frac{230}{146}x\right)^2 \mathrm{d}x = \left(\frac{230}{146}\right)^2 \frac{1}{3}x^3\Big|_0^{146} = \frac{1}{3}230^2\cdot 146 \approx 2.57\cdot 10^6 \quad \mathrm{m}^3$$

and hence its weight is

$$\frac{1}{3}230^2\cdot 146\cdot 2.3 \approx 5.921\cdot 10^6 \quad \mathrm{t}.$$

(b) The volume of the pyramid was

$$\int_{73}^{146}\left(\frac{230}{146}x\right)^2 \mathrm{d}x = \left(\frac{230}{146}\right)^2 \frac{1}{3}x^3\Big|_{73}^{146} = \frac{7}{24}230^2\cdot 146 \quad \mathrm{m}^3$$

and hence its weight about 5.181 million tonnes.

(c) About $\frac{5.921\cdot 10^6}{2.5} \approx 2.369\cdot 10^6$, or 2.4 million, blocks of limestone.

(d) Recall that on earth's surface, one kg weighs about $9.81\ \mathrm{N}$. The work therefore is

$$\int_0^{146}\left(\frac{230}{146}x\right)^2 (146-x) 2300\cdot 9.81 \mathrm{d}x \approx 2.120\cdot 10^{12} \quad \mathrm{J}.$$

(e) The building takes

$$\frac{50\cdot 2.120\cdot 10^{12}}{2{,}000{,}000} \approx 5.30\cdot 10^7$$

days of a man's work. To complete the work in 20 years

$$\frac{5.30\cdot 10^7}{20\cdot 365} \approx 7260$$

workers would be needed.

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Differential equations

Example 50. The **differential equation** $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ is solved by $y(x) = e^x$. It is also solved by $y(x) = 0$ and $y(x) = 7e^x$. Its general solution is $y(x) = Ce^x$ where C can be any number.

Example 51. The **initial value problem** $\frac{\mathrm{d}y}{\mathrm{d}x} = y$, $y(0) = 1$ has the unique solution $y(x) = e^x$.

The fact that the exponential function solves these simple equations is at the root of why it is so important!

Example 52. The general solution to the differential equation (DE) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2$ is $y(x) = \frac{1}{3}x^3 + C$.

In general, computing the antiderivative of $f(x)$ is the same as solving the (very special) DE $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$.

Verifying if a function solves a DE

Given a function, we can always check whether it solves a DE!

We can just plug it into the DE and see if left and right side agree. This means that we can always check our work as well as that we can verify solutions generated by someone else (or a computer algebra system) even if we don't know the techniques for solving the DE.

Example 53. Consider the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$.

(a) Is $y(x) = \frac{1}{x}$ a solution?

(b) $y(x) = -\frac{1}{x}$ a solution?

(c) Is $y(x) = -\frac{1}{x+3}$ a solution?

(d) Is $y(x) = 0$ a solution?

(e) Is $y(x) = 1$ a solution?

Solution.

(a) We compute $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$. On the other hand, $y^2 = \frac{1}{x^2}$. Since $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y^2 are not equal, $y = \frac{1}{x}$ is not a solution.

(b) We compute $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2}$. Since $y^2 = \left(-\frac{1}{x}\right)^2 = \frac{1}{x^2}$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$. Hence, $y = -\frac{1}{x}$ is a solution.

(c) We compute $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x+3)^2}$. Since $y^2 = \frac{1}{(x+3)^2}$ as well, we conclude that $y = -\frac{1}{x+3}$ is another solution.

(d) Since $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $y^2 = 0$ as well, we again conclude that $y = 0$ is another solution.

(e) Since $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ while $y^2 = 1$, we conclude that $y = 1$ is not a solution.

Comment. We will solve this DE shortly and find that the general solution is $y(x) = -\frac{1}{x+C}$ (where the solution $y = 0$ corresponds to $C \to \infty$).

Example 54. Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) We compute $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) We compute $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

Separation of variables

The next example demonstrates the method of **separation of variables** to solve (a certain class of) differential equations.

Example 55. Let us solve the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$ by separation of variables.

Comment. Some of the next steps might feel questionable... However, as illustrated above, we can always verify afterwards that we indeed found a solution.

In the first step, we separate the variables, including the differentials $\mathrm{d}y$ and $\mathrm{d}x$:

$$\frac{1}{y^2}\,\mathrm{d}y = \mathrm{d}x$$

[If the DE is of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$, then we would separate it as $\frac{1}{h(y)}\,\mathrm{d}y = g(x)\,\mathrm{d}x$.]

We then integrate both sides and compute the indefinite integrals:

$$\begin{aligned} \int \frac{1}{y^2}\,\mathrm{d}y &= \int \mathrm{d}x \\ -\frac{1}{y} &= x + C \qquad \text{[we combine the two constants of integration into one]} \end{aligned}$$

If possible (like here) we solve the resulting equation for y:

$$y = -\frac{1}{x + C}$$

Example 56. Find the general solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + y)e^x$, $y > -1$, using separation of variables.

Solution. We separate variables to get $\frac{1}{1 + y}\,\mathrm{d}y = e^x\,\mathrm{d}x$.

Integrating both sides, we find $\ln(1 + y) = e^x + C$. (Since $y > -1$, we don't need to write $\ln|1 + y|$.)

We now solve for y to get $y(x) = e^{e^x + C} - 1$.

Exercise. As an exercise in differentiation, verify that $y(x)$ indeed solves the differential equation.

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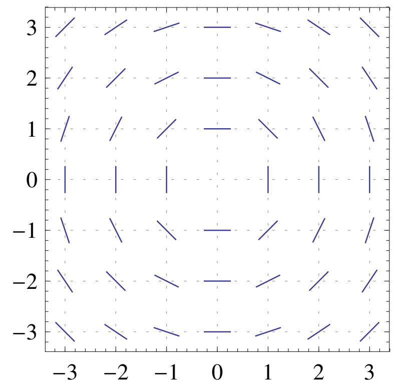
Slope fields, or sketching solutions to DEs

The next example illustrates that we can "plot" solutions to differential equations (it does not matter if we are able to actually solve them).

Comment. This is an important point because "plotting" really means that we can numerically approximate solutions. For complicated systems of differential equations, such as those used to model fluid flow, this is usually the best we can do. Nobody can actually solve these equations.

Example 57. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1, 2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1, 2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Separation of variables, cont'd

Example 58. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Example 59. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found earlier in Example 58, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

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Example 60. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = 2$.

Solution. Proceeding as in the previous example, we find $y(x) = \sqrt{4 - x^2}$.

Example 61. Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$, $y(0) = 3$.

Solution. We separate variables to get $\frac{1}{y}\mathrm{d}y = x\mathrm{d}x$.

Integrating both sides, we find $\ln(y) = \frac{1}{2}x^2 + C$. (Since $y = 3$ in the initial condition, we don't need to write $\ln|y|$ because we have $y > 0$ around the initial condition.)

We can then find C by using the values $x = 0$, $y = 3$ from the initial condition: $\ln(3) = \frac{1}{2} \cdot 0^2 + C$. So, $C = \ln(3)$.

We now solve $\ln(y) = \frac{1}{2}x^2 + \ln(3)$ for y to get $y(x) = e^{\frac{1}{2}x^2 + \ln(3)} = 3e^{\frac{1}{2}x^2}$.

Alternatively. We could have also first solved for y and then determined C with the same result.

Which differential equations can we actually solve using separation of variables?

- A general DE of first order is typically of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$.

  For instance, $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy) - x^2 y$.

  Comment. First order means that only the first derivative of y shows up. The most general form of a DE of first order is $F(x, y, y') = 0$ but we can usually solve for y' to get to the above form.

- The ones we can solve are **separable equations**, which are of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$ (although simple) is not separable.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y - x}$ is separable because we can write it as $\frac{\mathrm{d}y}{\mathrm{d}x} = e^y e^{-x}$.

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A very simple model of population growth

If $y(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{dy}{dt}$ might, from biological considerations, be (nearly) proportional to $y(t)$.

More down to earth, this is just saying "for a population 5 times as large, we expect 5 times as many babies".

Say, we have a population of $P=100$ and $P'=3$, meaning that the population changes by 3 individuals per unit of time. By how much do we expect a population of $P=500$ to change? (Think about it for a moment!)

Without further information, we would probably expect the population of $P=500$ to change by $5\cdot 3=15$ individuals per unit of time, so that $P'=15$ in that case. This is what it means for P' to be proportional to P. In formulas, it means that P'/P is constant or, equivalently, that $P'=kP$ for a proportionality constant k.

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}y}{\mathrm{d}t}=ky$ where k is the constant of proportionality.

The general solution to this DE is $y(t)=Ce^{kt}$. (Solve it yourself using separation of variables!)

Hence, mathematics tells us that populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 62. Let $y(t)$ describe the size of a population at time t. Suppose $y(0)=100$ and $y(1)=300$. Under the exponential model of population growth, find $y(t)$.

Solution. $y(t)$ solves the DE $\frac{\mathrm{d}y}{\mathrm{d}t}=ky$ and therefore is of the form $y(t)=Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k\cdot 0}=C=100$ and $Ce^{k}=100e^{k}=300$. Hence $k=\ln(3)$ and $y(t)=100e^{\ln(3)t}=100\cdot 3^{t}$.

Example 63. A yeast culture, with initial mass 12 g, is assumed to exhibit exponential growth. After 10 min, the mass is 15 g. What is the mass after $t\text{ min}$?

Solution. Let $y(t)$ be the mass in g after $t\text{ min}$. $y(t)$ solves the DE $\frac{\mathrm{d}y}{\mathrm{d}t}=ky$ and so is of the form $y(t)=Ce^{kt}$.

We now use that $y(0)=12$ and $y(10)=15$ to determine both C and k.

$Ce^{k\cdot 0}=C=12$ and $Ce^{10k}=12e^{10k}=15$. Hence $k=\frac{1}{10}\ln\left(\frac{5}{4}\right)$ and $y(t)=12e^{\frac{1}{10}\ln\left(\frac{5}{4}\right)t}=12\,(5/4)^{t/10}$.

Example 64. Just to give an indication of how the modelling can be refined, let us suppose we want to take limited resources into account, so that there is a maximum sustainable population size M. This situation could be modelled by the **logistic equation**

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ky\left(1-\frac{y}{M}\right).$$

Note that if y is small (compared to M), then $1-\frac{y}{M}\approx 1$, so $\frac{\mathrm{d}y}{\mathrm{d}t}\approx ky$, and we are back at our previous model. However, once the population is getting close to M then $1-\frac{y}{M}\approx 0$, so $\frac{\mathrm{d}y}{\mathrm{d}t}\approx 0$, which means that the population does not continue to grow.

Main challenge of modeling: A model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

Example 65. (review) Solve the initial value problem $\frac{dy}{dx} = xy$, $y(0) = 3$.

Solution. We separate variables to get $\frac{1}{y}dy = x dx$.

Integrating both sides, we find $\ln(y) = \frac{1}{2}x^2 + C$. (Since $y = 3$ in the initial condition, we don't need to write $\ln|y|$ because we have $y > 0$ around the initial condition.)

We can then find C by using the values $x = 0$, $y = 3$ from the initial condition: $\ln(3) = \frac{1}{2} \cdot 0^2 + C$. So, $C = \ln(3)$.

We now solve $\ln(y) = \frac{1}{2}x^2 + \ln(3)$ for y to get $y(x) = e^{\frac{1}{2}x^2 + \ln(3)} = 3e^{\frac{1}{2}x^2}$.

Alternatively. We could have also first solved for y and then determined C with the same result.

Logarithms

Review. $\ln(x) = \log_e(x)$ is the inverse function of e^x. In other words, for all real x,

$$\ln(e^x) = x.$$

Similarly, $e^{\ln(x)} = x$ for all $x > 0$.

Likewise. $\log_a(x)$ is the inverse function of a^x (where a is called the base).

- $\frac{d}{dx}\ln(x) = \frac{1}{x}$

  Why? Start with $e^{\ln(x)} = x$ and differentiate both sides to get $e^{\ln(x)} \cdot \ln'(x) = 1$. It therefore follows that $\ln'(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$.

  Note. We also have $\frac{d}{dx}\ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$ for $x < 0$. Together, this implies the next entry.

- $\int \frac{1}{x}dx = \ln|x| + C$

The following observation allows us to convert other bases to the **natural base** e:

(other bases) $a^x = e^{\ln(a)x}$ and $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

Why? $a^x = e^{\ln(a)x}$ follows be writing $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$. Can you see how $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ follows from this?

Alternatively, the defining property of $\log_a$ is that $\log_a(a^x) = x$. Because $\ln(a^x) = x\ln(a)$ we see that $f(x) = \frac{\ln(x)}{\ln(a)}$ also has the property that $f(a^x) = x$.

Example 66. Compute $\frac{d}{dx}\log_a(x)$.

Solution. It follows from $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ that $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$.

Example 67. Compute $\frac{d}{dx}2^x$ and $\int 2^x dx$.

Solution. Write $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$ to see that $\frac{d}{dx}2^x = \ln(2)\, e^{x\ln(2)} = \ln(2)\, 2^x$ and, likewise, $\int 2^x dx = \frac{1}{\ln(2)}e^{x\ln(2)} + C = \frac{2^x}{\ln(2)} + C$.

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Example 68. Rewrite $2^{\ln(x)}$ as a power of x.

Solution. We use the same "trick" as in the previous example (applying $e^{\ln(x)}$ to the expression) to get

$$2^{\ln(x)} = e^{\ln(2^{\ln(x)})} = e^{\ln(x)\ln(2)} = (e^{\ln(x)})^{\ln(2)} = x^{\ln(2)}.$$

Alternatively. Since $\ln(x) = \ln(2)\log_2(x)$, we have $2^{\ln(x)} = 2^{\ln(2)\log_2(x)} = (2^{\log_2(x)})^{\ln(2)} = x^{\ln(2)}$.

Example 69. Determine $\displaystyle\int \frac{2^{\ln(x)}}{x^2}\mathrm{d}x$.

Solution. By the previous example,

$$\int \frac{2^{\ln(x)}}{x^2}\mathrm{d}x = \int x^{\ln(2)-2}\mathrm{d}x = \frac{1}{\ln(2)-1}x^{\ln(2)-1} + C.$$

Example 70. Determine $\displaystyle\int \frac{(\ln x)^4}{3x}\mathrm{d}x$.

Solution. We substitute $u = \ln(x)$ in which case $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (so that the $\frac{1}{x}$ in the integrand will cancel out), to get

$$\int \frac{(\ln x)^4}{3x}\mathrm{d}x = \frac{1}{3}\int u^4\mathrm{d}u = \frac{1}{15}u^5 + C = \frac{1}{15}(\ln x)^4 + C.$$

Example 71. Determine $\displaystyle\int \frac{\ln(\ln x)}{x\ln x}\mathrm{d}x$.

Solution. We substitute $u = \ln(x)$ in which case $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (so that the $\frac{1}{x}$ in the integrand will cancel out). Since

$$\int \frac{\ln(\ln x)}{x\ln x}\mathrm{d}x = \int \frac{\ln u}{u}\mathrm{d}u.$$

In the new integral, we substitute $v = \ln(u)$ with $\mathrm{d}v = \frac{1}{u}\mathrm{d}u$ to get

$$\int \frac{\ln(\ln x)}{x\ln x}\mathrm{d}x = \int \frac{\ln u}{u}\mathrm{d}u = \int v\mathrm{d}v = \frac{1}{2}v^2 + C = \frac{1}{2}(\ln u)^2 + C = \frac{1}{2}(\ln(\ln x))^2 + C.$$

Example 72. Determine $\displaystyle\int_0^2 3^{-x}\mathrm{d}x$.

Solution. Write $3^{-x} = e^{\ln(3^{-x})} = e^{-x\ln(3)}$ to see that

$$\int_0^2 3^{-x}\mathrm{d}x = \left[-\frac{1}{\ln(3)}e^{-x\ln(3)}\right]_0^2 = \left[-\frac{1}{\ln(3)}3^{-x}\right]_0^2 = \frac{1}{\ln(3)}(1-3^{-2}) = \frac{8}{9\ln(3)}.$$

Example 73. Determine $\displaystyle\int \frac{\log_3(x)}{x}\mathrm{d}x$.

Solution. Since $\log_3(x) = \frac{\ln(x)}{\ln(3)}$, we find that $\displaystyle\int \frac{\log_3(x)}{x}\mathrm{d}x = \frac{1}{\ln(3)}\int \frac{\ln(x)}{x}\mathrm{d}x$.

We now substitute $u = \ln(x)$ in which case $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (so that the $\frac{1}{x}$ in the integrand will cancel out). Since

$$\int \frac{\ln(x)}{x}\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(x))^2 + C,$$

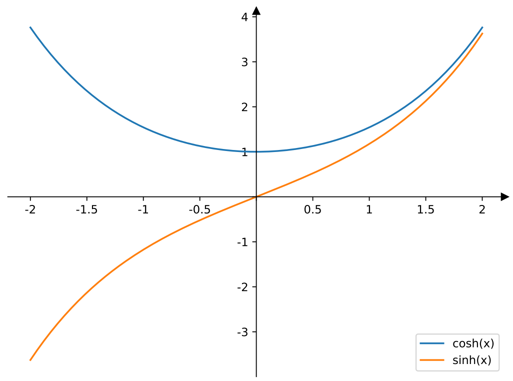
we get that $\displaystyle\int \frac{\log_3(x)}{x}\mathrm{d}x = \frac{1}{\ln(3)}\int \frac{\ln(x)}{x}\mathrm{d}x = \frac{1}{2\ln(3)}(\ln(x))^2 + B$ (where $B = \frac{C}{\ln(3)}$ is some constant).

Hyperbolic functions

The **hyperbolic cosine and sine** are $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

The remaining hyperbolic trigonometric functions are built from these two as expected.

For instance, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.



We will later see that $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$. For now observe and verify the following properties that reflect similar properties of $\cos$ and $\sin$:

- $\cosh'(x) = \sinh(x)$
  $\sinh'(x) = \cosh(x)$
- $\cosh(-x) = \cosh(x)$ (that is, $\cosh$ is an even function)
  $\sinh(-x) = -\sinh(x)$ (that is, $\sinh$ is an odd function)
- $\cosh^2(x) - \sinh^2(x) = 1$

  This property explains the name **hyperbolic functions**: the points $(x, y) = (\cosh(t), \sinh(t))$ produce the **unit hyperbola** $x^2 - y^2 = 1$. This is analogous to how cosine and sine parametrize the circle: in that case, the points $(x, y) = (\cos(t), \sin(t))$ produce the **unit circle** $x^2 + y^2 = 1$.

  Comment. Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  Comment. Plot the unit hyperbola. Then compare the graph to $y = \frac{1}{x}$. (This is a hyperbola, too!)

  Comment. Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry
- $e^x = \cosh(x) + \sinh(x)$

  This is a "cheap" version of **Euler's identity** $e^{ix} = \cos(x) + i\sin(x)$, which we will look at soon.
  In both cases, e^x and e^{ix} are broken up into their even part and odd part.

Example 74. Rewrite in terms of exponentials and simplify as much as possible:

(a) $4\sinh(\ln x)$

(b) $\cosh(3x) - \sinh(3x)$

Solution.

(a) $4\sinh(\ln x) = 4 \cdot \frac{e^{\ln x} - e^{-\ln x}}{2} = 2\left(x - \frac{1}{x}\right)$

(b) $\cosh(3x) - \sinh(3x) = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

Example 75. Determine the following:

(a) $\frac{\mathrm{d}}{\mathrm{d}x} 4\cosh(3x)$

(b) $\int 4\cosh(3x)\mathrm{d}x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(\sinh(x^2 + 3x))$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x} 4\cosh(3x) = 12\sinh(3x)$

(b) $\int 4\cosh(3x)\mathrm{d}x = \frac{4}{3}\sinh(3x) + C$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(\sinh(x^2 + 3x)) = \frac{1}{\sinh(x^2 + 3x)} \cdot \cosh(x^2 + 3x) \cdot 2x$

Since the hyperbolic functions are defined in terms of the exponential function, it is not surprising that their inverse functions can be expressed in terms of logarithms. We leave it at the following example.

Example 76. Express $\sinh^{-1}$ in terms of logarithms.

Solution. We start with $y = \sinh(x) = \frac{e^x - e^{-x}}{2}$ and need to solve for x.
Write $u = e^x$ so that the equation becomes $2y = u - \frac{1}{u}$.
Multiplying with u and rearranging, we obtain $u^2 - 2yu - 1 = 0$ which is a quadratic equation in u.
Using the quadratic formula, we find $u = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}$. Note that $u = e^x > 0$ so that we have to choose the $+$ sign here.
Since $u = e^x$, this implies $x = \ln(u) = \ln\left(y + \sqrt{y^2 + 1}\right)$.
In summary, we have found that $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Comment. It follows from $\sinh(x) = -i \sin(ix)$ that $\arcsin = \sin^{-1}$ can be similarly expressed in terms of logarithms. However, we will now have the imaginary i in that formula.

Example 77. Find the length of the curve $y = \cosh x$ from $x = -2$ to $x = 2$.

Solution. The length is

$$\begin{aligned} \int_{-2}^{2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x &= \int_{-2}^{2} \sqrt{1 + \sinh^2(x)} \mathrm{d}x = \int_{-2}^{2} \cosh(x) \mathrm{d}x \\ &= \left[\sinh(x)\right]_{-2}^{2} = \sinh(2) - \sinh(-2) = 2\sinh(2) = e^2 - e^{-2} \approx 7.254. \end{aligned}$$

Here, we used that $1 + \sinh^2(x) = \cosh^2(x)$ as we had observed earlier (in the form $\cosh^2(x) - \sinh^2(x) = 1$).
Also note that $\cosh(x) > 0$ so that we get $\sqrt{\cosh^2(x)} = +\cosh(x)$.

Spotlight on the exponential function

Euler's constant, the natural base

Euler's constant $e = 2.7182818284590452...$ is unavoidable in Calculus. For instance, starting with only division (which is all we need to define the function $1/x$), we obtain

$$\int \frac{1}{x}\mathrm{d}x = \log_e|x| + C.$$

Likewise, e^x is the only exponential whose derivative is itself. More professionally speaking, we have the following characterization of the exponential function:

(exponential function) e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x}2^x = \ln(2)\,2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.)
Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

The following is a **preview** of a series (infinite sum):

(preview of Taylor series) From the IVP above, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

This is the **Taylor series** for e^x at $x = 0$. More on these later!
Important note. We can indeed construct this infinite sum directly from $y' = y$ and $y(0) = 1$. To see this, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} = \frac{x^2}{2!}$.

Example 78. Suppose we have capital 1 and that, annually, we are receiving $1 = 100\%$ interest. How much capital do we have at the end of a year, if $\frac{1}{n}$ interest is paid n times a year?

[For instance, $n = 12$ if we receive monthly interest payments.]

Solution. At the end of the year, we have $\left(1 + \frac{1}{n}\right)^n$.

For instance. Here are a few values spelled out:

$$\begin{aligned}
n = 1{:} &\quad \left(1 + \frac{1}{n}\right)^n = 2\\
n = 4{:} &\quad \left(1 + \frac{1}{n}\right)^n = 2.4414...\\
n = 12{:} &\quad \left(1 + \frac{1}{n}\right)^n = 2.6130...\\
n = 100{:} &\quad \left(1 + \frac{1}{n}\right)^n = 2.7048...\\
n = 365{:} &\quad \left(1 + \frac{1}{n}\right)^n = 2.7145...\\
n = 1000{:} &\quad \left(1 + \frac{1}{n}\right)^n = 2.7169...\\
n \to \infty{:} &\quad \left(1 + \frac{1}{n}\right)^n \to e = 2.71828...
\end{aligned}$$

It is natural to wonder what happens if interest payments are made more and more frequently. As the entry for $n \to \infty$ shows, if we keep increasing n, then we will get closer and closer to $e = 2.7182818284590452...$ in our bank account after one year.

Challenge. Can you evaluate the limit $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ using your Calculus I skills?

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Euler's identity

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z = x + iy$ is x and we write $\mathrm{Re}(z) = x$.

  Likewise the **imaginary part** is $\mathrm{Im}(z) = y$.

  Observe that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 79. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

Comment. It follows that $\cos(x) = \mathrm{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \mathrm{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
In particular, we see from here that $\cos(x) = \cosh(ix)$ and $i\sin(x) = \sinh(ix)$ (or, equivalently, $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$).

Example 80. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

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Example 81. Which trig identity hides behind $e^{i(x+y)} = e^{ix}e^{iy}$?

Solution. We observe that

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i\sin(x+y) \\ e^{ix}e^{iy} &= [\cos(x) + i\sin(x)][\cos(y) + i\sin(y)] \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)). \end{aligned}$$

Comparing real and imaginary parts, we conclude that

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and
- $\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$.

Example 82. Which trig identity hides behind $e^{ix}\,e^{-ix} = 1$?

Solution. Note that

$$\begin{aligned} e^{ix}\,e^{-ix} &= [\cos(x) + i\sin(x)][\cos(-x) + i\sin(-x)] = [\cos(x) + i\sin(x)][\cos(x) - i\sin(x)] \\ &= \cos^2 x + \sin^2 x. \end{aligned}$$

Hence, $e^{ix}\,e^{-ix} = 1$ translates into Pythagoras' identity $\cos^2 x + \sin^2 x = 1$.

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Review. (from midterm exam) Here are two ways to determine $\int \frac{\mathrm{d}x}{2x}$. Which is correct?

(a) $\int \frac{\mathrm{d}x}{2x} = \frac{1}{2}\int \frac{\mathrm{d}x}{x} = \frac{1}{2}\ln|x| + C$

(b) We substitute $u = 2x$ (so that $\mathrm{d}u = 2\mathrm{d}x$) to get:

$\int \frac{\mathrm{d}x}{2x} = \int \frac{\frac{1}{2}\mathrm{d}u}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x| + C$

Solution. Both are correct! The answers look different but they only differ by a constant because

$$\frac{1}{2}\ln|2x| = \frac{1}{2}\ln(2|x|) = \frac{1}{2}(\ln(2) + \ln|x|).$$

Integration by parts

If we integrate both sides of the product rule, we obtain the following:

$$\begin{aligned} (fg)' &= f'g + fg' \\ \overset{\text{antiderivative}}{\rightsquigarrow} \quad f(x)g(x) &= \int f'(x)g(x)\mathrm{d}x + \int f(x)g'(x)\mathrm{d}x \end{aligned}$$

If we then solve for one of the two integrals, we get:

(integration by parts)

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

The following shorthand is very common as well:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Here, $u = f(x)$, $v = g(x)$ so that $\mathrm{d}u = f'(x)\mathrm{d}x$ and $\mathrm{d}v = g'(x)\mathrm{d}x$.

Example 83. Determine $\int x\cos(x)\,\mathrm{d}x$.

Solution. We choose $f(x) = x$ and $g'(x) = \cos(x)$, so that $g(x) = \sin(x)$ (note that we are free to choose the simplest antiderivative for $g(x)$), to get

$$\int x\cos(x)\,\mathrm{d}x = x\sin(x) - \int 1\cdot\sin(x)\,\mathrm{d}x = x\sin(x) + \cos(x) + C.$$

Example 84. Determine $\int x e^x\,\mathrm{d}x$.

Solution. We choose $f(x) = x$ and $g'(x) = e^x$, so that $g(x) = e^x$, to get

$$\int x e^x\,\mathrm{d}x = x e^x - \int 1\cdot e^x\,\mathrm{d}x = x e^x - e^x + C = (x-1)e^x + C.$$

Example 85. Determine $\int \ln(x)\,\mathrm{d}x$.

Solution. We choose $f(x) = \ln(x)$ and $g'(x) = 1$, so that $g(x) = x$, to get

$$\int \ln(x) \cdot 1\,\mathrm{d}x = \ln(x) \cdot x - \int \frac{1}{x} \cdot x\,\mathrm{d}x = x\ln(x) - x + C.$$

Example 86. Substitute $u = \ln(x)$ in the previous integral. What do you get?

Solution. If $u = \ln(x)$ then $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ so that $\mathrm{d}x = x\,\mathrm{d}u = e^u\mathrm{d}u$ (in the last step, we used that $x = e^u$).

We therefore get $\int \ln(x)\,\mathrm{d}x = \int u e^u\,\mathrm{d}u$.

By Example 84 we know $\int u e^u\,\mathrm{d}u = (u-1)e^u + C$ so that

$$\int \ln(x)\,\mathrm{d}x = \int u e^u\,\mathrm{d}u = (u-1)e^u + C = (\ln(x) - 1)e^{\ln(x)} + C = (\ln(x) - 1)x + C,$$

which matches what we obtained in Example 85.

Comment. We can also start by writing $x = e^u$ so that we immediately get $\mathrm{d}x = e^u\mathrm{d}u$. It depends on the integral, which of the two approaches is algebraically simpler.

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Review. By integrating the product rule, we get the formula for integration by parts:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x.$$

Example 87. (again) Determine $\int x e^x \,\mathrm{d}x$.

- To evaluate the integral, we should choose $f(x) = x$ and $g'(x) = e^x$, because $f'(x)$ becomes easier while $g(x)$ stays the same. Do it!
- If, on the other hand, we decide to choose $f(x) = e^x$ and $g'(x) = x$, then we obtain

  $$\int x e^x \,\mathrm{d}x = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x \,\mathrm{d}x.$$

  While certainly correct, we actually ended up with a more difficult integral.
  [On the other hand, because we know $\int x e^x \,\mathrm{d}x$, this means we now also know $\int x^2 e^x \,\mathrm{d}x$.]

Example 88. Determine $\int_0^1 x^2 e^{3x} \,\mathrm{d}x$.

Solution. We do integration by parts with $f(x) = x^2$ and $g'(x) = e^{3x}$ so that $f'(x) = 2x$ and $g(x) = \frac{1}{3}e^{3x}$ to get

$$\int x^2 e^{3x} \,\mathrm{d}x = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} \,\mathrm{d}x.$$

We now do integration by parts again with $f(x) = x$ and $g'(x) = e^{3x}$ so that $f'(x) = 1$ and $g(x) = \frac{1}{3}e^{3x}$ to get

$$\int x e^{3x} \,\mathrm{d}x = \frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x} \,\mathrm{d}x = \frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x}.$$

Taken together, this means

$$\int x^2 e^{3x} \,\mathrm{d}x = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x}\right) = \left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x}.$$

In particular,

$$\int_0^1 x^2 e^{3x} \,\mathrm{d}x = \left[\left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x}\right]_0^1 = \left(\frac{1}{3} - \frac{2}{9} + \frac{2}{27}\right)e^3 - \frac{2}{27} = \frac{5}{27}e^3 - \frac{2}{27}.$$

Alternatively. While doing integration by parts, we can carry the bounds along. For instance, for the first step,

$$\int_0^1 x^2 e^{3x} \,\mathrm{d}x = \left[\frac{1}{3}x^2 e^{3x}\right]_0^1 - \frac{2}{3}\int_0^1 x e^{3x} \,\mathrm{d}x = \frac{1}{3}e^3 - \frac{2}{3}\int_0^1 x e^{3x} \,\mathrm{d}x.$$

For practice, do the second step likewise to get the same final answer as before!

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Example 89. Determine $\int e^x\cos(x)\,\mathrm{d}x$.

Solution. We will need to integrate by parts twice. First, let $f(x)=\cos(x)$ and $g'(x)=e^x$ so that $f'(x)=-\sin(x)$ and $g(x)=e^x$ (the other way around works as well—see below!) to get

$$\int e^x\cos(x)\,\mathrm{d}x = e^x\cos(x) + \int e^x\sin(x)\,\mathrm{d}x.$$

The new integral is of the same level of difficulty, so it might seem like we haven't gained anything. But don't give up yet! Instead, integrate by parts again with $f(x)=\sin(x)$ and $g'(x)=e^x$ to arrive at

$$\int e^x\cos(x)\,\mathrm{d}x = e^x\cos(x) + \int e^x\sin(x)\,\mathrm{d}x = e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\,\mathrm{d}x.$$

We can now solve for $\int e^x\cos(x)\,\mathrm{d}x$ and find $\int e^x\cos(x)\,\mathrm{d}x = \frac{1}{2}(e^x\sin(x) + e^x\cos(x))$.

Solution. (variation) We proceed as before but now let $f(x)=e^x$ and $g'(x)=\cos(x)$ to get

$$\int e^x\cos(x)\,\mathrm{d}x = e^x\sin(x) - \int e^x\sin(x)\,\mathrm{d}x.$$

Again, we once more integrate by parts: choosing $f(x)=e^x$ and $g'(x)=\sin(x)$, we arrive at

$$\int e^x\cos(x)\,\mathrm{d}x = e^x\sin(x) - \int e^x\sin(x)\,\mathrm{d}x = e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,\mathrm{d}x.$$

As before, we can then solve for $\int e^x\cos(x)\,\mathrm{d}x$ to find $\int e^x\cos(x)\,\mathrm{d}x = \frac{1}{2}(e^x\sin(x) + e^x\cos(x))$.

We next discuss integrals of products of trig functions. The following is an example that we are already familiar with:

Example 90. (review/preview) $\int \sin^\lambda(x)\cos(x)\,\mathrm{d}x$ (with $\lambda\neq -1$)

Solution. We substitute $u=\sin(x)$, because then $\mathrm{d}u=\cos(x)\,\mathrm{d}x$, to get

$$\int \sin^\lambda(x)\cos(x)\,\mathrm{d}x = \int u^\lambda \mathrm{d}u = \frac{1}{\lambda+1}u^{\lambda+1} + C = \frac{\sin^{\lambda+1}(x)}{\lambda+1} + C.$$

Review. Recall that $\mathrm{d}g(x) = g'(x)\mathrm{d}x$ (since $\frac{\mathrm{d}}{\mathrm{d}x}g(x) = g'(x)$). Integration by parts therefore is often written as

$$\int f(x)\mathrm{d}g(x) = f(x)g(x) - \int g(x)\mathrm{d}f(x), \quad \text{or} \quad \int u\mathrm{d}v = uv - \int v\mathrm{d}u.$$

In the latter short form, we have set $u = f(x)$ and $v = g(x)$.

Trigonometric integrals

The following example illustrates that we somtimes have choices when integrating:

Example 91. $\int \sin(x)\cos(x)\,\mathrm{d}x$

Solution. (integration by parts) Integrating by parts with $f(x) = \sin(x)$, $g'(x) = \cos(x)$, $g(x) = \sin(x)$, we get

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \sin^2(x) - \int \cos(x)\sin(x)\,\mathrm{d}x,$$

from which we conclude that $\int \sin(x)\cos(x)\,\mathrm{d}x = \frac{1}{2}\sin^2(x) + C$.

Solution. (substitution) Substitute $u = \sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

Solution. (trig identity) Since $\sin(2x) = 2\cos(x)\sin(x)$, we have

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \frac{1}{2}\int \sin(2x)\mathrm{d}x = -\frac{1}{4}\cos(2x) + C.$$

Important comment. Note that $\frac{1}{2}\sin^2(x) \neq -\frac{1}{4}\cos(2x)$ (for instance, plug in $x = 0$ to see that). However, the two functions are indeed equal up to a constant (namely, $\frac{1}{2}\sin^2(x) = -\frac{1}{4}\cos(2x) + \frac{1}{4}$) as we can see from the trig identity $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

Example 92. $\int \sin^m(x)\cos^3(x)\,\mathrm{d}x$ (with $m \neq -1, -3$)

Solution. We substitute $u = \sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\begin{aligned}\int \sin^m(x)\cos^3(x)\,\mathrm{d}x &= \int u^m\cos^2(x)\,\mathrm{d}u = \int u^m(1 - \sin^2(x))\,\mathrm{d}u = \int u^m(1 - u^2)\mathrm{d}u \\ &= \frac{u^{m+1}}{m+1} - \frac{u^{m+3}}{m+3} + C = \frac{\sin^{m+1}(x)}{m+1} - \frac{\sin^{m+3}(x)}{m+3} + C.\end{aligned}$$

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The strategy in the previous problem works whenever we have an odd power of cosine:

Example 93. Describe how we can determine $\int \sin^m(x)\cos^{2k+1}(x)\,\mathrm{d}x$.

Solution. Again, we substitute $u = \sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\int \sin^m(x)\cos^{2k+1}(x)\,\mathrm{d}x = \int u^m\cos^{2k}(x)\,\mathrm{d}u = \int u^m(1-\sin^2(x))^k\,\mathrm{d}u = \int u^m(1-u^2)^k\mathrm{d}u.$$

For a given integer $k \geqslant 0$, we can now multiply out the term $(1-u^2)^k$. Each resulting term (after multiplying with u^m) can then be integrated using the power rule (as in the previous example).

Extrapolating this strategy, we can integrate the following products of trigonometric function:

- $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$, with $n = 2k+1$ odd, can be evaluated by substituting $u = \sin(x)$.

  See previous example!

- $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$, with m odd, can be likewise evaluated by substituting $u = \cos(x)$.

- $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$, with both m, n even, can be reduced via

  $$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}.$$

  [Then, multiply out the integrand. The resulting integrals have smaller exponents, and we (recursively) apply our strategy to each of them (if the $2x$ bothers you, substitute $u = 2x$).]

Example 94. Determine $\int \cos^2(x)\mathrm{d}x$.

Solution. (trig identity) Since both exponents are even (the exponent of $\sin(x)$ is 0, which is even), we use the trig identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$:

$$\int \cos^2(x)\mathrm{d}x = \int \frac{1+\cos(2x)}{2}\mathrm{d}x = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$$

Solution. (integration by parts—only for practice) We choose $f(x) = \cos(x)$ and $g'(x) = \cos(x)$ (so that $g(x) = \sin(x)$) to get

$$\int \cos^2(x)\mathrm{d}x = \cos(x)\sin(x) + \int \sin^2(x)\mathrm{d}x = \cos(x)\sin(x) + \int (1-\cos^2(x))\mathrm{d}x.$$

Note that our integral appears on both sides. Solving for it, we conclude that

$$\int \cos^2(x)\mathrm{d}x = \frac{1}{2}(\cos(x)\sin(x) + x) + C.$$

Our final answer looks different at first glance but is the same because $\sin(2x) = 2\cos(x)\sin(x)$.

Example 95. Determine $\int \cos^2(x)\sin^2(x)\mathrm{d}x$.

Solution. Since both exponents are even, we use the trig identities $\cos^2(x) = \frac{1+\cos(2x)}{2}$, $\sin^2(x) = \frac{1-\cos(2x)}{2}$:

$$\int \cos^2(x)\sin^2(x)\mathrm{d}x = \int \frac{1+\cos(2x)}{2}\cdot\frac{1-\cos(2x)}{2}\mathrm{d}x = \frac{1}{4}\int (1-\cos^2(2x))\mathrm{d}x = \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)\mathrm{d}x.$$

We now use $\cos^2(x) = \frac{1+\cos(2x)}{2}$ again (or we could use Example 94) to find

$$\int \cos^2(2x)\mathrm{d}x = \int \frac{1+\cos(4x)}{2}\mathrm{d}x = \frac{1}{2}x + \frac{1}{8}\sin(4x) + B.$$

Combined, we have (we rename the constant of integration to absorb the factor of $-1/4$)

$$\int \cos^2(x)\sin^2(x)\mathrm{d}x = \frac{1}{4}x - \frac{1}{4}\left(\frac{1}{2}x + \frac{1}{8}\sin(4x)\right) + C = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C.$$

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One exponent may also be negative (in the next example, we integrate $[\sin(x)]^1\,[\cos(x)]^{-1}$).

Example 96. Determine $\int \tan(x)\,\mathrm{d}x$.

Solution. $\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\mathrm{d}x$, so we substitute $u = \cos(x)$ (then $\mathrm{d}u = -\sin(x)\mathrm{d}x$) to get

$$\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\mathrm{d}x = -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

Solution. (harder—only for practice) For some exercise in substituting, we can also substitute $u = \sin(x)$ (but can you explain how we can tell beforehand that $u = \cos(x)$ should be the better choice?). Then $\mathrm{d}u = \cos(x)\mathrm{d}x$ or, equivalently, $\mathrm{d}x = \frac{1}{\cos(x)}\mathrm{d}u$, so that we get

$$\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\mathrm{d}x = \int \frac{u}{\cos^2(x)}\mathrm{d}u = \int \frac{u}{1-\sin^2(x)}\mathrm{d}u = \int \frac{u}{1-u^2}\mathrm{d}u.$$

We now substitute $v = 1 - u^2$ (so that $\mathrm{d}v = -2u\mathrm{d}u$) to get

$$\begin{aligned}\int \tan(x)\,\mathrm{d}x &= \int \frac{u}{1-u^2}\mathrm{d}u = -\frac{1}{2}\int \frac{\mathrm{d}v}{v} = -\frac{1}{2}\ln|v| + C = -\frac{1}{2}\ln|1-u^2| + C\\ &= -\frac{1}{2}\ln|1-\sin^2(x)| + C = -\frac{1}{2}\ln|\cos^2(x)| + C = -\ln|\cos(x)| + C\end{aligned}$$

as earlier.

Trigonometric substitutions

Example 97. Everybody knows that $\cos^2 x + \sin^2 x = 1$.
Divide both sides by $\cos^2 x$ to find $1 + \tan^2 x = \sec^2 x$.

Likewise, dividing by $\sin^2 x$, we find $\cot^2 x + 1 = \csc^2(x)$. However, note that in this identity we cannot have $x = 0$.

if you see	**try substituting**	because
$a^2 - x^2$ (especially $\sqrt{a^2 - x^2}$)	$x = a\sin\theta$	$a^2 - (a\sin\theta)^2 = a^2\cos^2\theta$
$a^2 + x^2$ (especially $\sqrt{a^2 + x^2}$)	$x = a\tan\theta$	$a^2 + (a\tan\theta)^2 = a^2\sec^2\theta = \frac{a^2}{\cos^2\theta}$
and, somewhat less importantly:		
$x^2 - a^2$ (especially $\sqrt{x^2 - a^2}$)	$x = a\sec\theta$	$(a\sec\theta)^2 - a^2 = a^2\tan^2\theta$

Note that (by completing the square and doing a simple linear substitution), you can put any quadratic term $ax^2 + bx + c$ into one of these three cases (for instance, $x^2 + 2x + 3 = (x+1)^2 + 2 = u^2 + 2$ with the simple linear substitution $u = x + 1$).
This is why trigonometric substitution occurs frequently for certain kinds of integrals.

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Example 98. Determine $\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$.

Solution. We substitute $x=\sin\theta$ (with $\theta\in[-\pi/2,\pi/2]$ so that $\theta=\arcsin(x)$) because then $1-x^2=\cos^2\theta$. Since $\mathrm{d}x=\cos\theta\,\mathrm{d}\theta$, we find

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}}=\int \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{1-\sin^2\theta}}=\int \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{\cos^2\theta}}=\int 1\,\mathrm{d}\theta=\theta+C=\arcsin(x)+C.$$

[Note that in order to conclude $\sqrt{\cos^2\theta}=\cos\theta$, we used that $\theta\in[-\pi/2,\pi/2]$ and that $\cos\theta\geqslant 0$ for these values of θ.]

On the other hand. Let's compute the derivative of $\arcsin(x)$ directly from its definition as the inverse function of $\sin(x)$: take the derivative of both sides of $\sin(\arcsin(x))=x$ to get $\cos(\arcsin(x))\arcsin'(x)=1$. Hence

$$\arcsin'(x)=\frac{1}{\cos(\arcsin(x))}=\frac{1}{\sqrt{1-\sin^2(\arcsin(x))}}=\frac{1}{\sqrt{1-x^2}}.$$

Comment. Because the role of $\cos$ and $\sin$ in $\cos^2\theta+\sin^2\theta=1$ is symmetric, we can also substitute $x=\cos\theta$. Do it! When comparing final answers, keep in mind that

$$\arccos(x)=\frac{\pi}{2}-\arcsin(x).$$

This reflects the relationship $\cos(\theta)=\sin\left(\frac{\pi}{2}-\theta\right)$.

Example 99. Determine $\int \sqrt{1-x^2}\,\mathrm{d}x$.

Solution. We substitute $x=\sin\theta$ (with $\theta\in(-\pi/2,\pi/2)$ so that $\theta=\arcsin(x)$) because then $1-x^2=\cos^2\theta$. Since $\mathrm{d}x=\cos\theta\,\mathrm{d}\theta$, we find

$$\int \sqrt{1-x^2}\,\mathrm{d}x=\int \cos^2\theta\,\mathrm{d}\theta=\text{...by parts...}=\frac{1}{2}(\cos(\theta)\sin(\theta)+\theta)+C=\frac{1}{2}\left(x\sqrt{1-x^2}+\arcsin x\right)+C.$$

See Example 94 for the integration by parts. In the final step, we used $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$ (instead of $\cos(\arcsin(x))$).

Comment. Note that $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ is the area of a quarter of the unit circle (and so has to be $\pi/4$). Using the antiderivative we just computed, we indeed find (since $\sin(\pi/2)=1$ we have $\arcsin(1)=\pi/2$)

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x=\left[\frac{\arcsin x+x\sqrt{1-x^2}}{2}\right]_0^1=\frac{\frac{\pi}{2}+0}{2}-\frac{0+0}{2}=\frac{\pi}{4}.$$

Example 100. Determine $\int \frac{1}{t^2\sqrt{t^2-4}}\,\mathrm{d}t$.

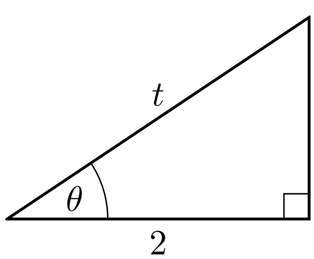
Solution. We substitute $t=2\sec\theta$ because then $t^2-4=4(\sec^2\theta-1)=4\tan^2\theta$.

Since $\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{\mathrm{d}}{\mathrm{d}\theta}2\sec\theta=2\sec\theta\tan\theta$ (you can work this out from $\sec\theta=\frac{1}{\cos\theta}$), we get

$$\int \frac{1}{t^2\sqrt{t^2-4}}\,\mathrm{d}t=\int \frac{1}{4\sec^2\theta\sqrt{4\tan^2\theta}}2\sec\theta\tan\theta\,\mathrm{d}\theta=\frac{1}{4}\int \frac{1}{\sec\theta}\mathrm{d}\theta=\frac{1}{4}\int \cos\theta\,\mathrm{d}\theta=\frac{1}{4}\sin\theta+C.$$

Our final step consists in simplifying $\sin\theta$ given that $t=2\sec\theta$.

For this, draw a right-angled triangle with angle θ. To encode the relationship $\sec\theta=\frac{\text{hyp}}{\text{adj}}=\frac{t}{2}$, we assign the hypothenuse length t and the adjacent side length 2 as in the diagram to the right.



By Pythagoras, the opposite side then has length $\sqrt{t^2-4}$. It follows that

$$\sin\theta=\frac{\text{opp}}{\text{hyp}}=\frac{\sqrt{t^2-4}}{t}.$$

Overall, we have therefore found that

$$\int \frac{1}{t^2\sqrt{t^2-4}}\,\mathrm{d}t=\frac{\sqrt{t^2-4}}{4t}+C.$$

Example 101. Determine $\int \frac{1}{1+x^2}\,\mathrm{d}x$.

Solution. Of course, we already know that $\int \frac{1}{1+x^2}\mathrm{d}x=\arctan(x)+C$. On the other hand, in the alternative solution below, we pretend that we didn't.

Solution. We substitute $x=\tan\theta$ because then $1+x^2=\sec^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta=\sec^2\theta$, we get

$$\int \frac{1}{1+x^2}\,\mathrm{d}x=\int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sec^2\theta}=\int \mathrm{d}\theta=\theta+C=\arctan(x)+C.$$

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Example 102. Determine $\int \frac{1}{(1+x^2)^2}\mathrm{d}x$. [That's an integral we care about for partial fractions!]

Solution. We substitute $x=\tan\theta$ because then $1+x^2=\sec^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta=\sec^2\theta$, we get

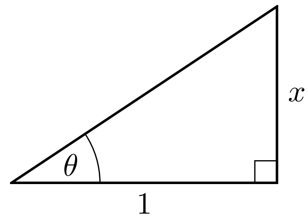
$$\int \frac{1}{(1+x^2)^2}\mathrm{d}x=\int \frac{\sec^2\theta\mathrm{d}\theta}{(\sec^2\theta)^2}=\int \frac{\mathrm{d}\theta}{\sec^2\theta}=\int \cos^2\theta\mathrm{d}\theta.$$

From Example 94, we know that

$$\int \cos^2\theta\mathrm{d}\theta=\frac{1}{2}(\cos(\theta)\sin(\theta)+\theta)+C.$$

After replacing $\theta=\arctan(x)$, we could stop here, except that our answer can be considerable simplified!

For this, draw a right-angled triangle with angle θ. To encode the relationship $\tan\theta=\frac{\mathrm{opp}}{\mathrm{adj}}=x$, we assign the opposite side length x and the adjacent side length 1 as in the diagram to the right.



By Pythagoras, the hypothenuse then has length $\sqrt{1+x^2}$. It follows that

$$\cos\theta=\frac{\mathrm{adj}}{\mathrm{hyp}}=\frac{1}{\sqrt{1+x^2}},\quad \sin\theta=\frac{\mathrm{opp}}{\mathrm{hyp}}=\frac{x}{\sqrt{1+x^2}}.$$

Hence $\cos(\theta)\sin(\theta)=\frac{x}{1+x^2}$ so that, combined, we get

$$\int \frac{1}{(1+x^2)^2}\mathrm{d}x=\frac{1}{2}(\cos(\theta)\sin(\theta)+\theta)+C=\frac{1}{2}\left[\frac{x}{1+x^2}+\arctan(x)\right]+C.$$

Comment. We just showed that, for instance, $\sin(\arctan(x))=\frac{x}{\sqrt{1+x^2}}$.

Partial fractions

Review. $\text{rational function} = \dfrac{\text{polynomial}}{\text{another polynomial}}$

Example 103. We are surely all familiar with putting stuff on a common demoninator like in

$$\frac{2}{x+1} + \frac{3}{x-1} = \frac{2(x-1)+3(x+1)}{(x+1)(x-1)} = \frac{5x+1}{(x+1)(x-1)}.$$

Partial fractions refers to reversing this process of putting things on a common denominator.

Example 104. The previous example allows us to easily compute the following integral:

$$\int \frac{5x+1}{(x+1)(x-1)} \mathrm{d}x = \int \frac{2}{x+1} \mathrm{d}x + \int \frac{3}{x-1} \mathrm{d}x = 2\ln|x+1| + 3\ln|x-1| + C.$$

Make sure that you are comfortable with integrating the two simpler integrals!
[For instance, notice that substituting $u = x+1$ in the first of the two, we have $\mathrm{d}u = \mathrm{d}x$ which is why we just get $\log$ of $x+1$.]

Example 105. Evaluate $\displaystyle\int \frac{x+4}{x(x-2)} \mathrm{d}x$ by partial fractions.

Solution. Partial fractions tells us that $\dfrac{x+4}{x(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-2}$ for some numbers A, B that we still need to find:

- To find A and B we multiply both sides with $x(x-2)$ to clear denominators:

  $$x+4 = (x-2)A + xB$$

- This equation has to be true for all values of x:
  Set $x=0$ to get $4 = -2A$ so that $A = -2$.
  Set $x=2$ to get $6 = 2B$ so that $B = 3$.

- We can now verify that, indeed, $\dfrac{x+4}{x(x-2)} = \dfrac{-2}{x} + \dfrac{3}{x-2}$.

We therefore have $\displaystyle\int \frac{x+4}{x(x-2)} \mathrm{d}x = \int \frac{-2}{x} \mathrm{d}x + \int \frac{3}{x-2} \mathrm{d}x = -2\ln|x| + 3\ln|x-2| + C.$

Important. Setting $x=0$ and $x=2$ makes our life particularly easy. On the other hand, note that we can set x to any values to get valid equations for A and B (once we have two equations for those two unknowns, we should be able to solve for them).
Alternatively, note that both sides of $x+4 = (x-2)A + xB$ are polynomials in x. We can therefore also equate coefficients: comparing the coefficients of x gives $1 = A + B$ while comparing constant coefficients gives $4 = -2A$. Solving these, we again find $A = -2$ and $B = 3$.

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Example 106. Evaluate $\int \frac{2x-5}{x(x+1)(x+2)}\mathrm{d}x$ by partial fractions.

Solution. Partial fractions tells us that $\frac{2x-5}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$ for certain numbers A, B, C.

- We multiply both sides with $x(x+1)(x+2)$ to clear denominators:

$$2x-5 = (x+1)(x+2)A + x(x+2)B + x(x+1)C$$

- Set $x=0$ to get $-5 = 2A$ so that $A = -\frac{5}{2}$.
  Set $x=-1$ to get $-7 = -B$ so that $B = 7$.
  Set $x=-2$ to get $-9 = 2C$ so that $C = -\frac{9}{2}$.

- Consequently: $\frac{2x-5}{x(x+1)(x+2)} = -\frac{5}{2}\cdot\frac{1}{x} + \frac{7}{x+1} - \frac{9}{2}\cdot\frac{1}{x+2}$.

Hence: $\int \frac{2x-5}{x(x+1)(x+2)}\mathrm{d}x = -\frac{5}{2}\int\frac{\mathrm{d}x}{x} + 7\int\frac{\mathrm{d}x}{x+1} - \frac{9}{2}\int\frac{\mathrm{d}x}{x+2} = -\frac{5}{2}\ln|x| + 7\ln|x+1| - \frac{9}{2}\ln|x+2| + C$.

To decompose the rational function $\frac{f(x)}{g(x)}$ into partial fractions:

(a) Check that degree $f(x) <$ degree $g(x)$. (Otherwise, long division!)

(b) Factor $g(x)$ as far as possible.

(c) For each factor of $g(x)$ collect terms as follows:

- For a linear factor $x-r$, occuring as $(x-r)^m$ in $g(x)$, these terms are

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \ldots + \frac{A_m}{(x-r)^m}.$$

- For a quadratic factor x^2+px+q, occuring as $(x^2+px+q)^m$ in $g(x)$, these terms are

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \ldots + \frac{B_mx+C_m}{(x^2+px+q)^m}.$$

(d) Determine the values of the unknown constants (the A's, B's and C's).

Example 107. Evaluate $\int \frac{x^4+3x^3+1}{x^3-x}\mathrm{d}x$.

Solution. (outline) In order to proceed as in the previous problem, we need to address two things:

(a) Since the degree of the numerator is not less than the degree of the denominator, we need to first perform long division. In this case, we get

$$\frac{x^4+3x^3+1}{x^3-x} = x+3+\frac{x^2+3x+1}{x^3-x}.$$

(b) Factor the denominator: $x^3-x = x(x^2-1) = x(x-1)(x+1)$.

Therefore,

$$\int \frac{x^4+3x^3+1}{x^3-x}\mathrm{d}x = \int\left(x+3+\frac{x^2+3x+1}{x(x-1)(x+1)}\right)\mathrm{d}x = \frac{1}{2}x^2+3x+\int\frac{x^2+3x+1}{x(x-1)(x+1)}\mathrm{d}x.$$

We then can evaluate the final integral as in the previous problem.

Example 108. Evaluate $\int \frac{x^4+3x^3+1}{x^3-x}\,\mathrm{d}x$.

Solution.

- Since the degree of the numerator is not less than the degree of the denominator, we first perform long division. In this case, we get
$$\frac{x^4+3x^3+1}{x^3-x} = x+3+\frac{x^2+3x+1}{x^3-x}.$$
- Factor the denominator: $x^3-x = x(x^2-1) = x(x-1)(x+1)$.
- $\frac{x^4+3x^3+1}{x^3-x} = x+3+\frac{x^2+3x+1}{x(x-1)(x+1)}$
- By partial fractions $\frac{x^2+3x+1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$ for certain numbers A, B, C.
  - We multiply both sides with $x(x-1)(x+1)$ to clear denominators:
  $$x^2+3x+1 = (x-1)(x+1)A + x(x+1)B + x(x-1)C$$
  - Set $x=0$ to get $1=-A$ so that $A=-1$.
  Set $x=1$ to get $5=2B$ so that $B=\frac{5}{2}$.
  Set $x=-1$ to get $-1=2C$ so that $C=-\frac{1}{2}$.

Therefore,

$$\begin{aligned}\int \frac{x^4+3x^3+1}{x^3-x}\,\mathrm{d}x &= \int\left(x+3-\frac{1}{x}+\frac{5/2}{x-1}-\frac{1/2}{x+1}\right)\mathrm{d}x \\ &= \frac{1}{2}x^2+3x-\ln|x|+\frac{5}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|.\end{aligned}$$

Example 109. Determine the shape (but not the exact numbers involved) of the partial fraction decomposition of the following rational functions.

(a) $\frac{x^2-2}{x^4-x^2}$

(b) $\frac{x^7-2}{x^4-x^2}$

(c) $\frac{x^3-7x+1}{x^2(x^2+1)}$

(d) $\frac{x^2+5}{(x+2)^3(x^2+1)^2}$

Solution.

(a) $\frac{x^2-2}{x^4-x^2} = \frac{x^2-2}{x^2(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}$

(b) Note that in this case, we need to do long division first. Since $x^7/x^4 = x^3$, the result is of the form Ax^3+Bx^2+Cx+D with some remainder that still needs to be divided by x^4-x^2. Hence:
$\frac{x^7-2}{x^4-x^2} = \frac{x^7-2}{x^2(x-1)(x+1)} = Ax^3+Bx^2+Cx+D+\frac{E}{x}+\frac{F}{x^2}+\frac{G}{x-1}+\frac{H}{x+1}$

(c) $\frac{x^3-7x+1}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$

(d) $\frac{x^2+5}{(x+2)^3(x^2+1)^2} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}$

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Example 110. Evaluate $\int \frac{x^7+3x+1}{x^4+x^2}\,\mathrm{d}x$.

Solution.

- Since the degree of the numerator is not less than the degree of the denominator, we first perform long division. In this case, we get

$$\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{x^3+3x+1}{x^4+x^2}.$$

- For the remainder part, partial fractions now tells us its decomposed shape:

$$\frac{x^3+3x+1}{x^4+x^2} = \frac{x^3+3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

  - We multiply both sides with $x^2(x^2+1)$ to clear denominators:

$$x^3+3x+1 = x(x^2+1)A + (x^2+1)B + x^2(Cx+D)$$

  - We can now compare the coefficients of $x^3, x^2, x, 1$ on both sides.
    Coefficients of x^3: $1 = A + C$
    Coefficients of x^2: $0 = B + D$
    Coefficients of x: $3 = A$
    Coefficients of 1: $1 = B$.
    Hence, $A = 3$, $B = 1$, $C = 1 - A = -2$, $D = -B = -1$.
    Note. By coefficient of 1 we mean the constant terms of the polynomials (the stuff without any x).
    Alternatively. We can also plug in values for x to get equations in A, B, C, D. Unfortunately, our only "magic" choice is $x = 0$. This gives $B = 1$. Instead of plugging in random values for x (we could do that!) we can then subtract the $(x^2+1)B$ from both sides and divide by x to get the simpler $x^2 - x + 3 = (x^2+1)A + x(Cx+D)$. Then we can again set $x = 0$ to find $3 = A$. Finish this for practice!

  - Consequently: $\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{3}{x} + \frac{1}{x^2} + \frac{-2x-1}{x^2+1}$.

Finally, we can integrate to find:

$$\begin{aligned}\int \frac{x^7+3x+1}{x^4+x^2}\,\mathrm{d}x &= \int \left(x^3 - x + \frac{3}{x} + \frac{1}{x^2} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right)\mathrm{d}x \\ &= \frac{1}{4}x^4 - \frac{1}{2}x^2 + 3\ln|x| - \frac{1}{x} - \ln(x^2+1) - \arctan(x) + C\end{aligned}$$

Here, we computed $\int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln(x^2+1) + C$ by substituting $u = x^2+1$. Do it!

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Example 111. Evaluate $\displaystyle\int \frac{2x+1}{x^2+6x+9}\mathrm{d}x$.

- This time, after factoring, partial fractions tells us that

$$\frac{2x+1}{x^2+6x+9} = \frac{2x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

- Clearing denominators, $2x+1=(x+3)A+B$. Setting $x=-3$, we find $-5=B$. There is no "magic" next choice for x so we just set $x=0$ (any other choice works as well) to get $1=3A-5$, which implies $A=2$.
- Integration now is again straightforward (make sure it is to you!):

$$\int \frac{2x+1}{x^2+6x+9}\mathrm{d}x = \int \frac{2}{x+3}\mathrm{d}x + \int \frac{-5}{(x+3)^2}\mathrm{d}x = 2\ln|x+3| + \frac{5}{x+3} + C.$$

Comment. While there is no "magic" next choice for x, we can take derivatives to get rid of B! Indeed, differentiating $2x+1=(x+3)A+B$ gives $2=A$ directly.

Comment. The form $\frac{A}{x+3}+\frac{B}{(x+3)^2}$ is equivalent to the form $\frac{Cx+D}{(x+3)^2}$. However, the former is more useful for integrating.

Improper integrals

Example 112. Determine $\displaystyle\int_0^\infty e^{-x}\mathrm{d}x$.

This integral is an example of an **improper integral** of type I (because one of its limits is ∞).

Make a sketch!

Solution. Replacing the upper limit with b, we have $\displaystyle\int_0^b e^{-x}\,\mathrm{d}x = \left[-e^{-x}\right]_0^b = 1-e^{-b}$.

Therefore, $\displaystyle\int_0^\infty e^{-x}\mathrm{d}x = \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}(1-e^{-b}) = 1$.

Solution. (short version) Once experienced, we can just write $\displaystyle\int_0^\infty e^{-x}\mathrm{d}x = \left[-e^{-x}\right]_0^\infty = 0-(-1) = 1$ with the understanding that we used $\displaystyle\lim_{x\to\infty} -e^{-x} = 0$.

Example 113. Determine $\displaystyle\int_1^\infty \frac{1}{x^4}\mathrm{d}x$ as well as $\displaystyle\int_1^\infty \frac{1}{x}\mathrm{d}x$.

Make a sketch! In the first quadrant, both functions look pretty similar.

Solution.

(a) $\displaystyle\int_1^\infty \frac{1}{x^4}\mathrm{d}x = \left[-\frac{1}{3x^3}\right]_1^\infty = 0-\left(-\frac{1}{3}\right) = \frac{1}{3}$ where we used that $\displaystyle\lim_{x\to\infty} -\frac{1}{3x^3} = 0$.

(b) $\displaystyle\int_1^\infty \frac{1}{x}\mathrm{d}x = \left[\ln|x|\right]_1^\infty$ but $\displaystyle\lim_{x\to\infty}\ln|x| = \infty$. We thus say that this integral **diverges** (to ∞ in this case).

Example 114. Determine $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x$.

Make a sketch! (For that, note that the integrand is always positive and that it goes to 0 as $x \to \infty$ or $x \to -\infty$. Also, you can see that the maximum occurs at $x=0$. Taken together, the graph looks like a single mound.)

Your final answer should be π.

Solution. Recall that $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan(x) + C$. If necessary, review $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and its inverse function $\arctan(x)$ to recall that

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}, \qquad \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}.$$

We therefore have $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x = \left[\arctan(x)\right]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

The following integral is an example of an improper integral of type II (because the integrand has a vertical asymptote at one of the limits).

Example 115. Determine $\int_0^1 \frac{1}{x} \mathrm{d}x$.

Make a sketch!

Solution. $\int_0^1 \frac{1}{x} \mathrm{d}x = \left[\ln|x|\right]_0^1$ but $\lim_{x\to 0^+} \ln|x| = -\infty$.

Thus, the integral diverges (to ∞, in this case).

Example 116. The following is VERY WRONG:

$$[\text{bad!}] \qquad \int_{-2}^{2} \frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_{-2}^{2} = -\frac{1}{3} - 1 = -\frac{4}{3} \qquad [\text{bad!}]$$

Note how even the answer is screaming trouble: we integrated something positive and got a negative value.

What went wrong? The integrand has a problem at $x=-1$!

To be precise, it has a vertical asymptote at $x=-1$. Since $\frac{1}{(x+1)^2}$ is not differentiable (not even continuous) at $x=-1$, we can only use the antiderivative $-\frac{1}{x+1}$ for $x \neq -1$. Since -1 is in the domain of integration $[-2,2]$, we cannot directly apply the Fundamental Theorem of Calculus to this integral.

Instead, we need to split the integral into two improper integrals and analyze these individually:

$$\int_{-2}^{2} \frac{\mathrm{d}x}{(x+1)^2} = \int_{-2}^{-1} \frac{\mathrm{d}x}{(x+1)^2} + \int_{-1}^{2} \frac{\mathrm{d}x}{(x+1)^2}.$$

But $\int_{-2}^{-1} \frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_{-2}^{-1}$ diverges because $\lim_{x\to-1^-} \left(-\frac{1}{x+1}\right) = \infty$. Hence, our original integral diverges.

[Note that, working on the second integral, the limit we encounter is $\lim_{x\to-1^+} \left(-\frac{1}{x+1}\right) = -\infty$. This integral, by itself, also diverges.]

Example 117. Does $\int_2^3 \frac{\mathrm{d}x}{x-2}$ converge? Does $\int_2^3 \frac{\mathrm{d}x}{\sqrt{x-2}}$ converge?

Solution. Final answers: No. Yes.

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Direct comparison and limit comparison

Sometimes we just want to know if an integral converges or diverges. In that case, we can compare the integrand with simpler functions.

The following illustrates our approach but is not meant to be exhaustive. The same ideas apply (suitably adjusted), for instance, when the functions are negative.

We assume that both $f(x)$ and $g(x)$ are continuous on $[a,\infty)$ and positive.

(direct comparison)

- $\int_a^\infty f(x)\mathrm{d}x$ converges if $0 \leqslant f(x) \leqslant g(x)$ on $[a,\infty)$ and $\int_a^\infty g(x)\mathrm{d}x$ converges.
- $\int_a^\infty f(x)\mathrm{d}x$ diverges if $f(x) \geqslant g(x) \geqslant 0$ on $[a,\infty)$ and $\int_a^\infty g(x)\mathrm{d}x$ diverges.

(limit comparison)

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ for $0 < L < \infty$ then
  $\int_a^\infty f(x)\mathrm{d}x$ converges if and only if $\int_a^\infty g(x)\mathrm{d}x$ converges.

Example 118. Determine whether the following integrals converge or diverge.

(a) $\int_1^\infty \frac{1}{x^5+2}\mathrm{d}x$

(b) $\int_2^\infty \frac{\sqrt{x^2+4}}{x^2}\mathrm{d}x$

(c) $\int_2^\infty \frac{\sqrt{x+4}}{x^2}\mathrm{d}x$

(d) $\int_2^\infty \frac{4-\sin(x)}{x^2}\mathrm{d}x$

(e) $\int_2^\infty \frac{4-\sin(x)}{x}\mathrm{d}x$

(f) $\int_1^\infty \frac{e^x}{x^2}\mathrm{d}x$

(g) $\int_1^\infty \frac{1}{\sqrt{e^{2x}+3x}}\mathrm{d}x$

Solution. The following are just indications of how to proceed. Fill in the details!

(a) We can apply the limit comparison test with $\frac{1}{x^5+2}$ and $\frac{1}{x^5}$ because $\lim_{x\to\infty} \frac{1/(x^5+2)}{1/x^5} = 1$.

Since $\int_1^\infty \frac{1}{x^5}\mathrm{d}x = \left[-\frac{1}{4x^4}\right]_1^\infty = \frac{1}{4}$ converges, it follows that $\int_1^\infty \frac{1}{x^5+2}\mathrm{d}x$ converges as well.

(b) Do limit comparison with $\frac{\sqrt{x^2}}{x^2} = \frac{1}{x}$ to conclude that this integral diverges.

(c) Do limit comparison with $\frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$ to conclude that this integral converges.

(d) Do a direct comparison with $\frac{5}{x^2}$ to conclude that this integral converges.

(e) Do a direct comparison with $\frac{3}{x}$ to conclude that this integral diverges.

(f) Note that $\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$. Hence the integral obviously diverges.

(g) Do limit comparison with $\frac{1}{\sqrt{e^{2x}}} = e^{-x}$ to conclude that this integral converges.

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Review. Direct comparison and limit comparison

L'Hospital's rule

Example 119. The following example illustrates that limits of the form "$\frac{\infty}{\infty}$" are completely undetermined. Anything is possible for the actual limit:

- $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$
- $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to\infty} \frac{x}{3x} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}$
- $\lim_{x\to\infty} \frac{x\,(1+\sin^2(x))}{x} = \lim_{x\to\infty} (1+\sin^2(x))$ This limit does not exist.

Theorem 120. (L'Hospital's rule) If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

The same conclusion holds if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

[It is important to realize that L'Hospital's rule only applies to the undetermined cases "$\frac{\infty}{\infty}$" and "$\frac{0}{0}$".]

Example 121. $\int_0^\infty x e^{-3x} \mathrm{d}x =$

Your final answer should be $\frac{1}{9}$.

Along the way, you will need the limit $\lim_{x\to\infty} x e^{-3x} = \lim_{x\to\infty} \frac{x}{e^{3x}} \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{3e^{3x}} = 0$.

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Sequences

A **sequence**, often denoted $\{a_n\}$, is an infinite list of its **terms** $a_1, a_2, a_3, \ldots$.
We'll define it precisely later, but one thing we are interested in is the **limit** $\lim_{n\to\infty} a_n$ (if it exists).
Here are a few first examples of sequences:

- $2, 4, 6, 8, 10, \ldots$ (that is, $a_1 = 2$, $a_2 = 4$, ...)

  This is the sequence $\{a_n\}$ with $a_n = 2n$. Clearly, $\lim_{n\to\infty} a_n = \infty$.

- $1, -1, 1, -1, 1, \ldots$

  This is the sequence $\{a_n\}$ with $a_n = (-1)^{n-1}$. The limit $\lim_{n\to\infty} a_n$ does not exist.

  Comment. Some part of the sequence "goes to" 1 but another part to -1. There is no single value that all terms approach.

- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

  This is the sequence $\{a_n\}$ with $a_n = \frac{1}{2^n}$. Clearly, $\lim_{n\to\infty} a_n = 0$.

  Preview. We will learn later that the **series** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$ also converges and equals 1. Can you maybe already explain why this is the case?

- $3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$

  This is the sequence $\{a_n\}$ where a_n consists of the first n (decimal) digits of π. Clearly, $\lim_{n\to\infty} a_n = \pi$.

- $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

  This is the sequence $\{a_n\}$ with $a_n = \frac{1}{n^2}$. Clearly, $\lim_{n\to\infty} a_n = 0$.

  Preview. We will learn later that the **series** $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$ also converges and equals $\frac{\pi^2}{6}$.

- $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

  These are the **Fibonacci numbers** $\{F_n\}$. They are defined **recursively**: $F_n = F_{n-1} + F_{n-2}$ together with the **initial values** $F_1 = 1$, $F_2 = 1$. Clearly, $\lim_{n\to\infty} F_n = \infty$.

- $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots$

  These are quotients of Fibonacci numbers $\{a_n\}$ with $a_n = \frac{F_{n+1}}{F_n}$.

  Numerically, $1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, \ldots$ Looks like $\lim_{n\to\infty} a_n$ exists and is about 1.618.

Sequences and series. In a little bit, we will also be interested in **series**. These are infinite sums such as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$. Do not confuse these two!

Confusing sequences and series would be like confusing a function and its definite integral.

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Limits of sequences

If $\lim_{x\to\infty} f(x) = L$ (the limit of a function: x is real)

then $\lim_{n\to\infty} f(n) = L$ (the limit of a sequence: n is an integer).

Important. The reverse is not true: for instance, $\lim_{n\to\infty} \sin(\pi n) = 0$ but $\lim_{x\to\infty} \sin(\pi x)$ does not exist.

Example 122. Determine the following limits:

(a) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1}$

(b) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^3+n+1}$

(c) $\lim_{n\to\infty} \frac{e^{3n}}{n^2}$

(d) $\lim_{n\to\infty} \frac{\sin(n)}{n}$

(e) $\lim_{n\to\infty} \cos\left(\pi - \frac{1}{n^2}\right)$

Solution. In each example, try to first "see" the limit! Then, apply some technique (like L'Hospital) to confirm.

(a) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1} = \frac{3}{8}$

The easiest way to see this is to note that the main terms in the numerator and denominator are $3n^2$ and $8n^2$. It follows that the limit is the same as $\lim_{n\to\infty} \frac{3n^2}{8n^2} = \lim_{n\to\infty} \frac{3}{8} = 0$.

We can make this argument precise in two different ways:

- $\frac{3n^2+7n-8}{8n^2+n+1} = \frac{3+\frac{7}{n}-\frac{8}{n^2}}{8+\frac{1}{n}+\frac{1}{n^2}}$ and now we can observe that all terms like $7/n$ go to 0 as $n\to\infty$.
- Since the quotient is of the undetermined form "$\frac{\infty}{\infty}$", we can apply L'Hospital (twice):

$$\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{6n+7}{16n+1} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{6}{16} = \frac{3}{8}$$

(b) $\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^3+n+1} = 0$

(c) $\lim_{n\to\infty} \frac{e^{3n}}{n^2} = \infty$

This is clear if you keep in mind that exponential growth exceeds any polynomial growth.

If needed, we can apply L'Hospital (twice!) since the limit is of the form "$\frac{\infty}{\infty}$":

$$\lim_{n\to\infty} \frac{e^{3n}}{n^2} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{3e^{3n}}{2n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{9e^{3n}}{2} = \infty$$

(d) $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Note that $-\frac{1}{n} \leqslant \frac{\sin(n)}{n} \leqslant \frac{1}{n}$. Since our sequence is squeezed between two sequences which approach 0, our limit has to be 0 as well.

Important. We cannot apply L'Hospital because the limit is not of the form "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$". If we did, we would get the limit $\lim_{n\to\infty} \frac{\cos(n)}{1}$ which does not exist (because the values oscillate between -1 and 1).

(e) $\lim_{n\to\infty} \cos\left(\pi - \frac{1}{n^2}\right) = \cos(\pi) = -1$

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Review. The following forms are indeterminate: "$\frac{\infty}{\infty}$", "$\frac{0}{0}$", "$0\cdot\infty$", "∞^0", "1^∞", "0^0". By applying $\ln$ in the last three cases, we can always write these as "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$", so that we can apply L'Hospital.

Example 123. Determine the following limits:

(a) $\lim_{n\to\infty} \frac{\ln n}{n}$

(b) $\lim_{n\to\infty} n^{1/n}$

(c) $\lim_{n\to\infty} \sqrt[n]{\pi^2}$

(d) $\lim_{n\to\infty} \sqrt[n]{n^2}$

(e) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n$

Solution.

(a) $\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$

We were able to use L'Hospital because the limit was of the form "$\frac{\infty}{\infty}$".

(b) $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp(\ln(n^{1/n})) = \lim_{n\to\infty} \exp\left(\frac{1}{n}\ln(n)\right) = \exp(0) = 1$

Note that we used the limit from the previous part.

Comment. We wrote $\exp(x) = e^x$ simply to avoid using exponents for typographical reasons.

(c) $\lim_{n\to\infty} \sqrt[n]{\pi^2} = \lim_{n\to\infty} \pi^{2/n} = \pi^0 = 1$

(d) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \exp(\ln(n^{2/n})) = \lim_{n\to\infty} \exp\left(\frac{2}{n}\ln(n)\right) = \exp(0) = 1$

(e) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty} \exp\left(\ln\left(\left(1+\frac{2}{n}\right)^n\right)\right) = \lim_{n\to\infty} \exp\left(n\ln\left(1+\frac{2}{n}\right)\right) = e^2$

In the final step, we used that $\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{2}{n}}\cdot\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}} = 2.$

Comment. More generally, $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for any x.

The following is a precise definition of the limit of a sequence:

Definition 124. $\lim_{n\to\infty} a_n = L$ means that:
for every $\varepsilon > 0$ there is a value N such that, for all $n > N$, $|a_n - L| < \varepsilon$.

Here are a few basic facts about limits:

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$
- If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then $\lim_{n\to\infty} (a_n + b_n) = A + B$ and $\lim_{n\to\infty} (a_n b_n) = AB$.
- If $\lim_{n\to\infty} a_n = A$ then $\lim_{n\to\infty} f(a_n) = f(A)$ provided that $f(x)$ is continuous at A.

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Example 125. $\lim_{n\to\infty} x^n = \begin{cases} \infty, & \text{if } x > 1, \\ 1, & \text{if } x = 1, \\ 0, & \text{if } -1 < x < 1, \\ \text{does not exist}, & \text{if } x \leqslant -1. \end{cases}$

If you think of a representative case for each situation, then this (important!) example becomes very natural:

- $\lim_{n\to\infty} 2^n =$
- $\lim_{n\to\infty} 1^n =$
- $\lim_{n\to\infty} (1/2)^n =$
- $\lim_{n\to\infty} (-1/2)^n =$
- $\lim_{n\to\infty} (-1)^n =$
- $\lim_{n\to\infty} (-2)^n =$

The geometric sum

(geometric sum)

$$\sum_{n=0}^{M} x^n = 1 + x + x^2 + \ldots + x^M = \frac{1 - x^{M+1}}{1 - x}$$

Why? Let us write $S = 1 + x + x^2 + \ldots + x^M$.
Note that $xS = x + x^2 + \ldots + x^M + x^{M+1}$ and that the result has most terms in common with our original sum. In fact, the right-hand side is $S - 1 + x^{M+1}$. This means that

$$xS = S - 1 + x^{M+1}.$$

Solving this for S, we find that $S = \frac{x^{M+1} - 1}{x - 1}$ which is equivalent to the above formula.

Example 126. Determine the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^M}$. What happens as $M \to \infty$?

Solution. This is a geometric sum with $x = \frac{1}{2}$. Thus,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^M} = \frac{1 - \left(\frac{1}{2}\right)^{M+1}}{1 - \frac{1}{2}}.$$

Note that

$$\lim_{M\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{M+1}}{1 - \frac{1}{2}} = \frac{1 - 0}{1 - \frac{1}{2}} = 2.$$

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Quiz. Four limits that we can "see" plus one that we need to work out, like the following:

$$\lim_{n\to\infty}\sqrt[n]{\frac{3}{n^2}}=\lim_{n\to\infty}\left(\frac{3}{n^2}\right)^{1/n}=\lim_{n\to\infty}\exp\left(\ln\left(\left(\frac{3}{n^2}\right)^{1/n}\right)\right)=\lim_{n\to\infty}\exp\left(\frac{1}{n}\ln\left(\frac{3}{n^2}\right)\right)=\exp(0)=1$$

where we used that

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{3}{n^2}\right)=\lim_{n\to\infty}\frac{\ln(3)-2\ln(n)}{n}\underset{\text{LH}}{\overset{``\frac{\infty}{\infty}"}{=}}\lim_{n\to\infty}\frac{-2\cdot\frac{1}{n}}{1}=0.$$

Series

A tortoise racing a Greek hero... **Zeno's paradox**:

https://en.wikipedia.org/wiki/Zeno%27s_paradoxes#Achilles_and_the_tortoise

Example 127. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$

Solution. Visual!

Solution. Redo this example by taking the limit of a geometric sum.

Geometric series

Review. The geometric sum is

$$\sum_{n=0}^{M}x^n=1+x+x^2+\ldots+x^M=\frac{1-x^{M+1}}{1-x}.$$

Taking the limit $M\to\infty$ in the geometric sum, we get: (recall that $\lim_{M\to\infty}x^M=0$ if $|x|<1$)

(geometric series) If $|x|<1$, then

$$\sum_{n=0}^{\infty}x^n=1+x+x^2+\ldots=\frac{1}{1-x}.$$

If $|x|\geqslant 1$, then the geometric series diverges.

Example 128. Compute the following series (or state that it diverges):

(a) $\sum_{n=0}^{\infty}\frac{1}{2^n}$

(b) $\sum_{n=3}^{\infty}\frac{1}{2^n}$

(c) $\sum_{n=0}^{\infty}\frac{7}{10^n}$

(d) $\sum_{n=2}^{\infty}\frac{7}{10^n}$

(e) $\sum_{n=0}^{\infty}\frac{5}{3^n}$

(f) $\sum_{n=2}^{\infty}3\cdot 4^{-n}$

(g) $\sum_{n=0}^{\infty}\left(\frac{7}{2^n}-\frac{3^n}{5^n}\right)$

(h) $\sum_{n=0}^{\infty}\frac{5^n}{3^n}$

(i) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

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Solution.

(a) $\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

(b) $\sum_{n=3}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2}\right) = 2 - \left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{4}$

(c) $\sum_{n=0}^{\infty} \frac{7}{10^n} = 7\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 7 \cdot \frac{1}{1-\frac{1}{10}} = \frac{70}{9}$

(d) $\sum_{n=2}^{\infty} \frac{7}{10^n} = \sum_{n=0}^{\infty} \frac{7}{10^n} - \left(\frac{7}{10^0} + \frac{7}{10^1}\right) = \frac{70}{9} - \left(7 + \frac{7}{10}\right) = \frac{7}{90}$

(e) $\sum_{n=0}^{\infty} \frac{5}{3^n} = 5\sum_{n=0}^{\infty} \frac{1}{3^n} = 5 \cdot \frac{1}{1-\frac{1}{3}} = \frac{15}{2}$

(f) $\sum_{n=2}^{\infty} 3 \cdot 4^{-n} = 3\sum_{n=2}^{\infty} \frac{1}{4^n} = 3\left(\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4}\right) = \frac{1}{4}$

(g) $\sum_{n=0}^{\infty} \left(\frac{7}{2^n} - \frac{3^n}{5^n}\right) = 7\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 7 \cdot \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{3}{5}} = 14 - \frac{5}{2} = \frac{23}{2}$

(h) $\sum_{n=0}^{\infty} \frac{5^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n$ doesn't converge because $\left|\frac{5}{3}\right| \geqslant 1$.

(i) $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ provided that $|-x^2| < 1$ (which is the same as $|x| < 1$). If this condition is not true, then the series diverges.

The very last example illustrates an important point. Namely, it shows that there is a novel way to think about (and get our hands on) functions like $\frac{1}{1+x^2}$.

Recall that we care about this function in particular, because it was a building block in partial fractions. For instance, we know that its antiderivative is $\arctan(x)$.

This is the main reason why we are learning about series in a course that focuses on functions!

We will see that it is very convenient to work with series representing functions: they can be differentiated and integrated, and give us an opportunity to work with functions that cannot be written in terms of the "usual" functions.

Example 129. Express the number $0.7777...$ as a rational number.

Solution. (using geometric series)

$$0.7777... = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + ... = \sum_{n=1}^{\infty} \frac{7}{10^n} = 7\left(\frac{1}{1-\frac{1}{10}} - 10^0\right) = 7\left(\frac{10}{9} - 1\right) = \frac{7}{9}$$

Solution. (highschool) Everyone is familiar with $0.3333... = \frac{1}{3}$. This implies that $0.1111... = \frac{1}{3} \cdot 0.3333... = \frac{1}{9}$. Hence, our number is $0.7777... = 7 \cdot 0.1111... = \frac{7}{9}$.

Example 130. (Halloween scare!) Let $a=b$. Then $a^2=ab$, so $a^2+a^2=a^2+ab$ or $2a^2=a^2+ab$. Hence, $2a^2-2ab=a^2-ab$ or $2(a^2-ab)=a^2-ab$. Cancelling, we arrive at $2=1$.

[Can you see the foul but disguised division by zero?!]

Example 131. Explain the following:

$$\frac{1}{499}=0.002004008016032064128256\ldots$$

Solution. The thing to explain is that there appear to be powers of 2 in the decimal on the right-hand side. Note that the right-hand side can be realized as a geometric series:

$$\frac{2}{1000}+\left(\frac{2}{1000}\right)^2+\left(\frac{2}{1000}\right)^3+\ldots=\frac{1}{1-\frac{2}{1000}}-1=\frac{1}{499}.$$

Example 132. (Halloween scare!) Foul play with divergent series:

$$\begin{aligned} 0 &= (1-1)+(1-1)+(1-1)+\ldots \\ &= 1-1+1-1+1-1+\ldots \\ &= 1+(-1+1)+(-1+1)+\ldots \\ &= 1+0+0\ldots \\ &= 1 \end{aligned}$$

Where did this go wrong? Note that in the second line we have the series $\sum_{n=0}^{\infty}(-1)^n$ which is divergent.

Lesson. Divergent series don't conform to our usual laws.

In other words, convergence of series is crucially important when working with them.

The fallacy above is somewhat similar to the argument "$\infty=\infty+1$, so $0=1$".

Example 133. (geometric series, again) To follow-up the previous example on a positive note: if a series converges, then we can work with it. The following argument for evaluating the geometric series is valid provided that the series converges (which we know it is if $|x|<1$):

$$S=1+x+x^2+x^3+\ldots \quad \rightsquigarrow \quad xS=x+x^2+x^3+x^4+\ldots=S-1 \quad \rightsquigarrow \quad S=\frac{1}{1-x}$$

Example 134. Write the series $e^{2x}+e^{3x}+e^{4x}+\ldots$ using Σ-notation and evaluate it.

Solution. $e^{2x}+e^{3x}+e^{4x}+\ldots=\sum_{n=2}^{\infty}e^{nx}=\sum_{n=0}^{\infty}e^{nx}-1-e^x=\frac{1}{1-e^x}-1-e^x$

Here, we need to assume that $|e^x|<1$ (or, equivalently, $x<0$).

Alternatively. $e^{2x}+e^{3x}+e^{4x}+\ldots=e^{2x}(1+e^x+e^{2x}+\ldots)=e^{2x}\sum_{n=0}^{\infty}e^{nx}=\frac{e^{2x}}{1-e^x}$

Check that this is the same!

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Example 135. Express the number $2.313131...$ as a rational number.

Solution.

$$2.313131... = 2 + \frac{31}{100} + \frac{31}{(100)^2} + \frac{31}{(100)^3} + ... = 2 + 31\sum_{n=1}^{\infty} \frac{1}{100^n} = 2 + 31\left(\frac{1}{1-\frac{1}{100}} - \frac{1}{100^0}\right) = 2 + \frac{31}{99} = \frac{229}{99}$$

This example plus the last one from previous class teach us something fundamental about numbers:

> Rational numbers are precisely those numbers which have a finite (like 1.5) or repeating (like $2.313131...$) decimal expansion.
>
> Moreover, there is some ambiguity because finite decimals, like 1.5, can also be written in the repeating fashion $1.5 = 1.4999....$

As a consequence, irrational numbers like $\sqrt{2}$ or π never have a repeating decimal expansion.

The nth-term test for divergence

Review. Recall that the improper integral $\int_N^\infty f(x)\mathrm{d}x$ converges if and only if the limit $\lim_{M\to\infty}\int_N^M f(x)\mathrm{d}x$ exists. Likewise, $\sum_{n=N}^{\infty} a_n$ converges if and only if the limit $\lim_{M\to\infty}\sum_{n=N}^{M} a_n$ exists.

For the series $\sum_{n=N}^{\infty} a_n$ to converge, it is necessary that $\lim_{n\to\infty} a_n = 0$.

Can you explain how this follows from the definition of limit?

Intuitively, this is simply saying that the only hope to be able to add infinitely many things (and get something finite) is if these things are very small.

> **Theorem 136. (nth-term test for divergence)** If $\lim_{n\to\infty} a_n$ is not 0, then $\sum_{n=1}^{\infty} a_n$ diverges.

(In particular, if the limit $\lim_{n\to\infty} a_n$ does not exist, then the series diverges.)

Example 137. Show that the following series all diverge.

(a) $\sum_{n=0}^{\infty} \frac{3^n+5^n}{10\cdot 5^n}$ (b) $\sum_{n=1}^{\infty} (-1)^n$ (c) $\sum_{n=1}^{\infty} \frac{n^2}{3n^2+7}$ (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\log(n)}$

Solution.

(a) Note that $\lim_{n\to\infty} \frac{3^n+5^n}{10\cdot 5^n} = \frac{1}{10} \neq 0$. Hence, the series diverges by the nth-term test for divergence.

(b) The sequence $(-1)^n$ does not converge to 0 as $n\to\infty$. Hence, the series diverges by the nth-term test.

(c) Since $\lim_{n\to\infty} \frac{n^2}{3n^2+7} = \frac{1}{3} \neq 0$ the series diverges by the nth-term test for divergence.

(d) This series diverges by the nth-term test for divergence because $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n}$ is not zero.

In fact, $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \infty$. (If you don't see this, apply L'Hospital!)

A word of caution. The nth-term test for divergence only gives a necessary condition. It is not sufficient!

For instance, as we will see next time, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges although $\lim_{n\to\infty} \frac{1}{n} = 0$.

Integral comparison test

Theorem 138. (Integral comparison test) Suppose that the function $f(x)$ is positive, continuous and decreasing for $x \geqslant N$. Then:

$$\sum_{n=N}^{\infty} f(n) \text{ converges} \iff \int_N^{\infty} f(x)\mathrm{d}x \text{ converges}$$

In other words, the series and integral both converge or both diverge.

Why? Make a sketch where you compare the area under the curve $f(x)$ with rectangles of width 1. (See Section 9.3 in our book for nice illustrations.)

Warning: if they converge, of course, the values of the series and the integral are going to be different!

Example 139. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges. Why?

Solution. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$, so we cannot directly use our test for divergence coming out of Theorem 136. However, we can combine terms as follows to see the divergence:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geqslant \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geqslant \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geqslant \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}} + \ldots$$

Solution. (integral comparison test) The function $f(x) = \frac{1}{x}$ is positive, continuous and decreasing for $x \geqslant 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ converges} \iff \int_1^{\infty} \frac{1}{x}\mathrm{d}x \text{ converges.}$$

Since $\int_1^M \frac{1}{x}\mathrm{d}x = \Big[\ln|x|\Big]_1^M = \ln M \to \infty$ as $M \to \infty$, the integral diverges. It follows, by comparison, that the harmonic series diverges, too.

Example 140. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

[It is considerably more difficult to show that, in fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

Solution. As in the previous example, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if and only if the integral $\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x$ converges. Since $\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x = \left[-\frac{1}{x}\right]_1^{\infty} = 0 - (-1) = 1$, the integral converges, and so the series converges as well.

More generally, we have the following result:

(p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a **p-series**. It converges if and only if $p > 1$.

Why? This follows from the integral comparison test, and because $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$. See next example.

Example 141. For what values of p does $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ converge?

Solution. For $p \neq 1$, we have $\int_1^\infty \frac{\mathrm{d}x}{x^p} = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^\infty$.

If $p > 1$, then $\lim_{x\to\infty} x^{-p+1} = \lim_{x\to\infty} \frac{1}{x^{p-1}} = 0$, and we find that the integral converges.

If $p < 1$, then $\lim_{x\to\infty} x^{-p+1} = \infty$, and we find that the integral diverges.

We are missing only the case $p = 1$: in that case, $\int_1^\infty \frac{1}{x}\mathrm{d}x = \left[\ln|x|\right]_1^\infty$ diverges because $\lim_{x\to\infty} \ln|x| = \infty$.

In summary, $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$.

Example 142. Determine whether the following series converge or diverge.

(a) $\sum_{n=0}^\infty \frac{1}{2n+1}$ (b) $\sum_{n=0}^\infty \frac{1}{n^2+1}$

Solution. You can use the integral comparison test to find that the first series diverges and the second series converges. However, it is more convenient to look at these series by comparison which is what we discuss next.

Direct comparison and limit comparison

Recall that we discussed direct comparison and limit comparison tests for improper integrals. The same ideas apply to series:

We assume that both $a_n \geqslant 0$ and $b_n \geqslant 0$.

(direct comparison)

- $\sum_{n=N}^\infty a_n$ converges if $a_n \leqslant b_n$ and $\sum_{n=N}^\infty b_n$ converges.
- $\sum_{n=N}^\infty a_n$ diverges if $a_n \geqslant b_n$ and $\sum_{n=N}^\infty b_n$ diverges.

(limit comparison)

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for $0 < L < \infty$ then $\sum_{n=N}^\infty a_n$ and $\sum_{n=N}^\infty b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=N}^\infty b_n$ converges, then $\sum_{n=N}^\infty a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=N}^\infty b_n$ diverges, then $\sum_{n=N}^\infty a_n$ diverges.

Example 143. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+\ln(n)}$ (b) $\sum_{n=1}^{\infty} \frac{n+\ln(n)}{n^2+4}$

Solution.

(a) **(direct comparison)** Note that, for all $n \geqslant 1$, $n^2+\log(n) \geqslant n^2$ and so $\frac{1}{n^2+\log(n)} \leqslant \frac{1}{n^2}$.

By comparison, $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} = \text{finite}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)}$ converges.

(limit comparison) Observe that we can just "see" this: for large n, our terms $\frac{1}{n^2+\log(n)}$ "behave" like $\frac{1}{n^2}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This reasoning is made precise by the limit comparison test. Spell out the details!

Comment. Like in this example, we often have a "natural" comparison with a geometric series or a p-series. In those cases, don't spend time thinking about the corresponding integral! Here, $\int_1^{\infty} \frac{dx}{x^2+\log(x)}$ is such that its antiderivative cannot even be written in terms of the functions we are familiar with.

(b) Our terms $a_n = \frac{n+\ln(n)}{n^2+4}$ behave like $\frac{n}{n^2} = \frac{1}{n}$ for large n. Thus, we should do limit comparison with $b_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n+\ln(n))}{n^2+4} = 1.$$

This means that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (this is the harmonic series), our series diverges as well.

Frequently, we have choices which test to apply to determine whether a series converges:

Example 144. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. (via integral comparison) The integral $\int_1^{\infty} \frac{\ln x}{x} dx = \int_0^{\infty} u du$ obviously diverges (note that we substituted $u = \ln x$), and hence $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges as well.

Comment. Usually, it is a good idea to avoid integral comparison unless there are no easier options.

Solution. (direct comparison) Note that $\frac{\log n}{n} > \frac{1}{n}$ for all $n > 3$ (because $\log n > 1$ for $n > 3$).

But already $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (note that $\sum_{n=3}^{\infty} \frac{\log n}{n} > \sum_{n=3}^{\infty} \frac{1}{n}$), so $\sum_{n=1}^{\infty} \frac{\log n}{n}$ has to diverge as well.

Solution. (limit comparison) We do limit comparison of $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$ so that $\frac{a_n}{b_n} = \ln n$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, we conclude that $\sum_{n=N}^{\infty} a_n$ diverges as well.

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Review. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Example 145. Determine whether the following series converge or diverge.

(a) $\sum_{n=0}^{\infty} \frac{1}{2n+1}$

(b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

(c) $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$

(d) $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{3n^2+3}$

Solution.

(a) We could do an integral comparison test (do it!) or we could do a direct comparison (do it!) but, when possible, it is easiest to do a limit comparison test. Namely, we can "see" that the terms $a_n = \frac{1}{2n+1}$ behave like $\frac{1}{2n}$ for large n.
Therefore, we do a limit comparison with $b_n = \frac{1}{n}$ (you could also choose $b_n = \frac{1}{2n}$ but that factor of 2 is not relevant):

$$\frac{a_n}{b_n} = \frac{n}{2n+1} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Since the limit is not zero and finite, limit comparison tells us that $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic, we know that it diverges. It follows that our series diverges as well.

(b) We proceed as in the previous part but now do a limit comparison with $b_n = \frac{1}{n^2}$ to conclude that our series converges (note that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p > 1$ and so converges).

(c) We do a limit comparison of the sequence $a_n = \frac{2n+1}{3n^2+3}$ with $b_n = \frac{1}{n}$.
First, we check that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3}$. By the limit comparison, we then find that $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(d) We do a limit comparison with $b_n = \frac{1}{n^{3/2}}$ to conclude that both series converge.

The ratio test

(absolute convergence)

We say that the series $\sum_{n=N}^{\infty} a_n$ **converges absolutely** if $\sum_{n=N}^{\infty} |a_n|$ converges.

It is not hard to see (check out Section 9.5 in the book) that absolute convergence implies (regular) convergence. It is often easier to work with series where all terms are $\geqslant 0$. Therefore, it is often easier to establish absolute convergence of a series (and then get convergence for free). Some tests like the **ratio test** even give us absolute convergence.

Caution! There are series which converge but which do not converge absolutely.

One example is the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$. We will discuss **alternating series** soon.

Note that a series $\sum_{n=N}^{\infty} a_n$ is geometric if $\frac{a_{n+1}}{a_n} = L$ is constant. It converges if and only if $|L| < 1$.

Theorem 146. (Ratio test) Suppose the limit $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

- If $L < 1$, then the series $\sum_{n=N}^{\infty} a_n$ converges (absolutely).
- If $L > 1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges.
- If $L = 1$, then we don't know. The test is inconclusive.

Example 147. Apply the ratio test to the geometric series $\sum_{n=0}^{\infty} x^n$.

Solution. In this case, $a_n = x^n$ and so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|$.

The ratio test with $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|$ then shows that $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$, and diverges if $|x| > 1$.

Important. The ratio test makes no statement about the cases $x = 1$ and $x = -1$. In these cases, we need to do additional analysis. Here, it is easy to see directly that the geometric series diverges when $x = 1$ or $x = -1$.

Example 148. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$

Solution.

(a) We apply the ratio test with $a_n = \frac{n^2}{2^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \frac{1}{2} \frac{(n+1)^2}{n^2} = \frac{1}{2} \frac{n^2 + 2n + 1}{n^2} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty$$

Since $\frac{1}{2} < 1$, the ratio test implies that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(b) Note that $\frac{2^n}{n^3} \rightarrow \infty \neq 0$. Hence, the series diverges.

Alternatively. Suppose we didn't realize this and, instead, we apply the ratio test with $a_n = \frac{2^n}{n^3}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^3} \frac{n^3}{2^n} = 2 \frac{n^3}{(n+1)^3} = 2 \frac{n^3}{n^3 + 3n^2 + 3n + 1} \rightarrow 2 \text{ as } n \rightarrow \infty$$

Since $2 > 1$, the ratio test implies that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges.

(c) We apply the ratio test with $a_n = \frac{(-1)^n 5^n}{n!}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} = \frac{5}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Since $0 < 1$, the ratio test implies that $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$ converges.

Review. Recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. This is the **factorial**.
It counts the number of ways in which you can order n objects.

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Review. Ratio test

The alternating series test

Theorem 149. (Alternating series test) If a_n is positive, decreasing with $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=N}^{\infty} (-1)^n a_n$ converges.

Why? Proof by picture!

Example 150. Does the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

Solution. Yes, it converges by the alternating series test: $a_n = \frac{1}{n}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$.

Important. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not **converge absolutely**.

Example 151. For which p does the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge? For which p does it converge absolutely?

Solution.

- If $p > 0$, then the series converges by the alternating series test, because $a_n = \frac{1}{n^p}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. If $p \leqslant 0$, then $\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ is not zero. Therefore, the series diverges.
- By definition, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Since this is just the usual p-series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $p > 1$.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$, and converges absolutely if and only if $p > 1$.

Power series

Definition 152. A **power series** (about $x = 0$) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

More generally, a **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Example 153. Determine all x for which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

Solution. We apply the ratio test with $a_n = \frac{x^n}{n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x|$ as $n \to \infty$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if $|x| < 1$ (and diverges if $|x| > 1$).

Note that the ratio test does not tell us what happens when $|x| = 1$. We need to look at those cases more carefully:

- $x = 1$: In this case we get the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges.
- $x = -1$: In this case we get the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which we know converges.

In summary, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if and only if x is in $[-1, 1)$.

Note. This is a power series around 0. The series converges for all x less than 1 away from 0. This is why we say that the power series has **radius of convergence** 1.

Looking ahead. Which function is hiding behind $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$?

Power series can be differentiated term-by-term and so $f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Therefore, $f(x) = \int \frac{1}{1-x} \mathrm{d}x = -\ln|1-x| + C$.

Finally, the fact that $f(0) = 0$ implies that $C = 0$. This means that $f(x) = -\ln|1-x|$.

Note that this function is nice at $x = 0$ (and around it) but that it has a problem for $x = 1$. Since converging power series do not have problems, this is one way to see that the radius of convergence is 1.

Let us again look at the special values $x = 1$ and $x = -1$. First, we clearly have $\lim_{x \to 1} f(x) = +\infty$.

On the other hand, $\lim_{x \to -1} f(x) = -\ln(2)$ implying that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots = -\ln(2) \approx -0.693$.

Theorem 154. Every power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a **radius of convergence** R, meaning:

(a) if $R = 0$, then the series converges only for $x = a$,

(b) if $0 < R < \infty$, then the series converges for all x such that $|x - a| < R$ but diverges if $|x - a| > R$ (in other words, R is as large as possible),

(c) if $R = \infty$, then the series converges for all x.

Note that, if $0 < R < \infty$, no general statement can be made for the case $|x - a| = R$.

The exact **interval of convergence** can be $(a - R, a + R)$ or $[a - R, a + R)$ or $(a - R, a + R]$ or $[a - R, a + R]$.

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Example 155. Determine the radius of convergence of the following power series and their exact interval of convergence.

(a) $\sum_{n=0}^{\infty} (n^2+4)\, x^n$ This is a power series about $x=0$.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} (x-3)^n$ This is a power series about $x=3$.

Solution.

(a) We apply the ratio test with $a_n = (n^2+4)\, x^n$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)^2+4)\, x^{n+1}}{(n^2+4)\, x^n}\right| = |x| \frac{n^2+2n+5}{n^2+4} \to |x|$ as $n\to\infty$

The ratio test implies that $\sum_{n=0}^{\infty} (n^2+4)\, x^n$ converges if $|x|<1$.

Thus the radius of convergence is 1.

The ratio test does not tell us what happens when $|x|=1$. We now look at those cases more carefully:

- $x=1$: $\sum_{n=0}^{\infty} (n^2+4)$ clearly diverges (because $\lim_{n\to\infty} (n^2+4)$ is not 0).
- $x=-1$: $\sum_{n=0}^{\infty} (n^2+4)(-1)^n$ clearly diverges (because $\lim_{n\to\infty} (n^2+4)(-1)^n$ is not 0).

Combined, $\sum_{n=0}^{\infty} (n^2+4)\, x^n$ converges if and only if x is in $(-1,1)$ (the exact interval of convergence).

(b) We apply the ratio test with $a_n = \frac{2^n}{\sqrt{n}}(x-3)^n$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{2^n(x-3)^n}\right| = 2|x-3| \sqrt{\frac{n}{n+1}} \to 2|x-3|$ as $n\to\infty$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} (x-3)^n$ converges if $|x-3|<\frac{1}{2}$.

So the radius of convergence is $\frac{1}{2}$.

The ratio test is inconclusive for $|x-3|=\frac{1}{2}$ or, equivalently, $x=3-\frac{1}{2}=\frac{5}{2}$ and $x=3+\frac{1}{2}=\frac{7}{2}$:

- $x=\frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test ($\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$).
  Comment. We could have also just recognized this as the alternating p-series with $p=\frac{1}{2}$.
- $x=\frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is the p-series with $p=\frac{1}{2}$ which diverges (because $p\leqslant 1$).

Combined, the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} (x-3)^n$ is $\left[\frac{5}{2}, \frac{7}{2}\right)$.

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Example 156. Determine the radius of convergence of $\sum_{n=1}^{\infty} \frac{5^n}{n^2}(4x-3)^{2n}$ (this is a power series about $\frac{3}{4}$) and its exact interval of convergence.

Solution. We apply the ratio test with $a_n = \frac{5^n}{n^2}(4x-3)^{2n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5^{n+1}}{(n+1)^2}(4x-3)^{2n+2}\frac{n^2}{5^n(4x-3)^{2n}}\right| = 5|4x-3|^2\frac{n^2}{(n+1)^2} \to 5|4x-3|^2$ as $n \to \infty$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{5^n}{n^2}(4x-3)^{2n}$ converges if $5|4x-3|^2 < 1$. To focus on x, we can rewrite this as $|4x-3| < \frac{1}{\sqrt{5}}$ or, equivalently, $\left|x - \frac{3}{4}\right| < \frac{1}{4\sqrt{5}}$.

In this latter form, we see that the radius of convergence is $\frac{1}{4\sqrt{5}}$.

The ratio test is inconclusive for $\left|x - \frac{3}{4}\right| = \frac{1}{4\sqrt{5}}$ or, equivalently, $x = \frac{3}{4} - \frac{1}{4\sqrt{5}}$ and $x = \frac{3}{4} + \frac{1}{4\sqrt{5}}$:

- $x = \frac{3}{4} + \frac{1}{4\sqrt{5}}$: $\sum_{n=1}^{\infty} \frac{5^n}{n^2}\left(\frac{1}{\sqrt{5}}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is the p-series with $p = 2$ which converges (because $p > 1$).
- $x = \frac{3}{4} - \frac{1}{4\sqrt{5}}$: $\sum_{n=1}^{\infty} \frac{5^n}{n^2}\left(-\frac{1}{\sqrt{5}}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is the same series and so converges as well.

Combined, the exact interval of convergence of $\sum_{n=1}^{\infty} \frac{5^n}{n^2}(4x-3)^{2n}$ is $\left[\frac{3}{4} - \frac{1}{4\sqrt{5}}, \frac{3}{4} + \frac{1}{4\sqrt{5}}\right]$.

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Example 157. What is the radius of convergence of the following power series?

(a) $\sum_{n=0}^{\infty} n! x^n$ (b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution.

(a) We apply the ratio test with $a_n = n! x^n$:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| = |x|(n+1) \rightarrow \infty$ as $n \rightarrow \infty$ (unless $|x| = 0$)

The ratio test implies that $\sum_{n=0}^{\infty} n! x^n$ diverges for all x except $x = 0$. The radius of convergence is 0.

(b) We again apply the ratio test, this time with $a_n = \frac{x^n}{n!}$:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n! x^{n+1}}{(n+1)! x^n}\right| = \frac{|x|}{n+1} \rightarrow 0$ as $n \rightarrow \infty$

The ratio test implies that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. The radius of convergence is ∞.

Comment. In Example 159 below, we find that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Power series as functions

The following simply states that we can treat power series as if they were polynomials of infinite degree when it comes to differentiating or integrating.

Theorem 158. (Term-by-term differentiation and integration)

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{on the interval } (a-R, a+R).$$

In this interval, $f(x)$ is arbitrarily often differentiable, and its derivatives can be obtained by differentiating the power series term by term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}, \end{aligned}$$

and so on. Likewise, $f(x)$ can be integrated term by term:

$$\int f(x) \mathrm{d}x = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

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Example 159. Compute the derivative of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Identify the function $f(x)$.

Solution. We have previously determined that this power series about $x=0$ has convergence radius $R=\infty$. Therefore, the function $f(x)$ is defined by the series for all x. Its derivative is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

This means that $f(x)$ satisfies the differential equation $y' = y$. It clearly also satisfies $y(0) = 1$. It follows that $f(x)$ is the unique solution of the IVP $y' = y$, $y(0) = 1$. We conclude that $f(x) = e^x$.

Example 160. Differentiate both sides of $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Solution. We find $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$. This identity is valid if $|x| < 1$ because that is the condition under which the geometric series converges.

Comment. If we prefer, we can also write $\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$.

Important observation. The new series $\sum_{n=1}^{\infty} n x^{n-1}$ has again radius of convergence 1 (like the geometric series).

This is a general phenomenon. Differentiating and integrating a power series does not change the radius of convergence. (However, this can change the behaviour at the endpoints of the interval of convergence.)
[Can you see this by thinking about the effect of an additional factor of n in a_n when applying the ratio test?]

Example 161. (extra) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution. The first series is just a geometric series: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1 = 1$

For the second series, we can use $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ with $x = \frac{1}{2}$. In that case, $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$.

Multiplying both sides by $\frac{1}{2}$, we obtain $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

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Taylor series

Review. Let $f(x)$ be a function. What is the best linear approximation to $f(x)$ at $x=a$?

Solution. The best linear approximation is the tangent line at $x=a$. This line has slope $f'(a)$ and goes through the point $(a, f(a))$. Hence, the equation for the best linear approximation is $f(a)+f'(a)(x-a)$.

Preview. We will see below that this is the **Taylor polynomial of order** 1. The Taylor polynomials of order M likewise provide the best approximations to $f(x)$ at $x=a$ using polynomials of degree up to M.

Comment. Of course, here we need to restrict to functions $f(x)$ that are differentiable at $x=a$.

The **Taylor series** of $f(x)$ at $x=a$ is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\ldots$$

- The **Taylor polynomial of order** M is the truncation $\sum_{n=0}^{M} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

  This is the best approximation of the function $f(x)$ at $x=a$ using a polynomial of degree up to M.

- If $f(x)$ can be written as a power series $x=a$ (nice functions can be!), then the Taylor series equals $f(x)$.

  Here, we are, of course, restricted to x within the interval of convergence.

- The Taylor series at $x=0$ is also called the **Maclaurin series** of $f(x)$.
- The functions we meet in practice can usually be written as power series (such functions are called **analytic** and are the fundamental object in complex analysis), at least about most points (and it usually is not difficult to tell if a special point is problematic).

  A theoretical guarantee is given by Taylor's formula, which says that

$$f(x)=\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x), \quad \text{with } R_N(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

  for some c between a and x. If $R_N(x)\to 0$ as $N\to\infty$, then $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Example 162. Determine the Taylor polynomials of order 2 and 3 for $f(x)=\frac{1}{x}$ at $x=3$.

Solution. The Taylor polynomial of order 2 is

$$f(3)+\frac{f'(3)}{1!}(x-3)+\frac{f''(3)}{2!}(x-3)^2=\frac{1}{3}-\frac{1}{9}(x-3)+\frac{1}{27}(x-3)^2.$$

Here, we used that $f'(x)=-\frac{1}{x^2}$ and $f''(x)=\frac{2}{x^3}$ so that $f'(3)=-\frac{1}{9}$ and $f''(3)=\frac{2}{27}$.

For order 3, we also compute $f'''(x)=-\frac{6}{x^4}$ so that $f'''(3)=-\frac{2}{27}$. Hence, the Taylor polynomial of order 3 is

$$f(3)+\frac{f'(3)}{1!}(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3=\frac{1}{3}-\frac{1}{9}(x-3)+\frac{1}{27}(x-3)^2-\frac{1}{81}(x-3)^3.$$

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Example 163. Determine the Taylor polynomial of order 3 for $f(x)=\sqrt{x}$ at $x=1$.

Solution. By definition, the Taylor polynomial in question is given by

$$\sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3.$$

Clearly $f(1)=1$ and we to compute the other values $f^{(n)}(1)$ as follows:

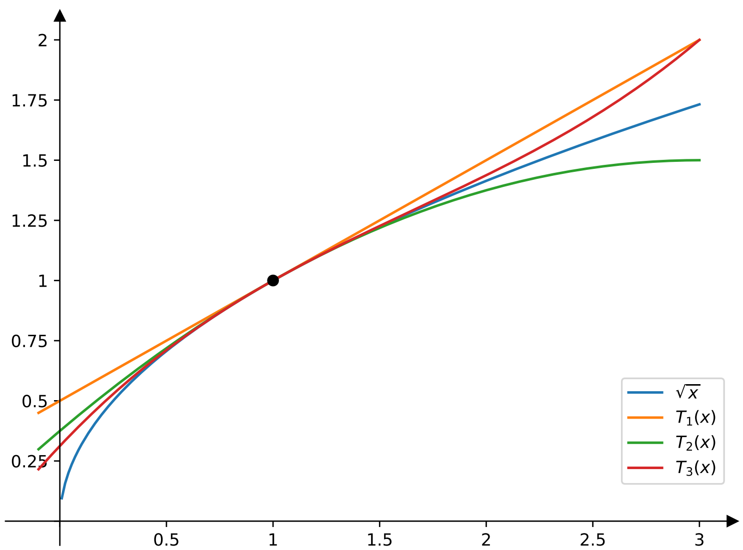
- $f'(x)=\frac{1}{2\sqrt{x}}$ so that $f'(1)=\frac{1}{2}$.
- $f''(x)=-\frac{1}{4x^{3/2}}$ so that $f''(1)=-\frac{1}{4}$.
- $f''(x)=\frac{3}{8x^{5/2}}$ so that $f'''(1)=\frac{3}{8}$.

The Taylor polynomial therefore is

$$f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{6}(x-1)^3 = 1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3.$$

Comment. Note that we can read off any Taylor polynomial $T_M(x)$ of lower order M by just truncating our polynomial. For instance, $T_2(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2$ and $T_1(x)=1+\frac{1}{2}(x-1)$.

The plot below shows how well these Taylor polynomials approximate $f(x)=\sqrt{x}$ near $x=1$.



Example 164.

(a) Determine the Taylor series of $f(x) = e^x$ at $x = 0$.

(b) Spell out the Taylor polynomial of order 4 for $f(x) = e^x$ at $x = 0$.

Solution.

(a) All derivatives of $f(x) = e^x$ are $f^{(n)}(x) = e^x$. In particular, $f^{(n)}(0) = 1$ for all n.
Therefore, the Taylor series of $f(x) = e^x$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

Comment. Take the derivative of the power series and observe how it reflects that $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$. Compare with Example 159.

Comment. We have that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ because e^x is "nice" and can be written as a power series. This can be justified, for instance, using Taylor's formula above.

(b) Truncating the Taylor series, the Taylor polynomial of order 4 is

$$\sum_{n=0}^{4} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

Example 165. Determine the Taylor series of $f(x) = e^{2x}$ at $x = 0$.

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Solution. Observe that $f^{(n)}(x) = 2^n e^{2x}$. Hence, $e^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

Note. $e^{2x} = e^x e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$

[For instance, we get the $\frac{4}{3}x^3$ as $1 \cdot \frac{x^3}{6} + x \cdot \frac{x^2}{2} + \frac{x^2}{2} \cdot x + \frac{x^3}{6} \cdot 1 = \frac{4}{3}x^3$.]

(Which matches the first terms of our series for e^{2x}.) This illustrates that we can multiply Taylor series.

Example 166. (review) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

The alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$.

For instance, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ ($p = 1/2$) converges. However, it does not converge absolutely.

Series that converge but don't converge absolutely are said to **converge conditionally**.

One has to be more careful with series that only converge conditionally. For instance, we cannot rearrange the order of the terms arbitrarily without affecting the overall sum.

Example 167.

(a) Determine the Taylor series of $f(x) = \cos(x)$ at $x = 0$.

(b) Spell out the Taylor polynomial of order 4 for $f(x) = \cos(x)$ at $x = 0$.

Solution.

(a) The derivatives of $f(x)$ cycle through $\cos(x), -\sin(x), -\cos(x), \sin(x), \ldots$.
In particular, the values $f^{(n)}(0)$ cycle through $1, 0, -1, 0, \ldots$.
That is, $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$.
Therefore, the Taylor series of $f(x) = \cos(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Note. Assuming that $\cos x$ can be written as a power series at $x = 0$, we conclude that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Again, this can be justified via Taylor's formula or a differential equation.

(b) Truncating the Taylor series, the Taylor polynomial of order 4 is

$$1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Example 168. Determine the Taylor series of $\int e^{-x^2} \mathrm{d}x$ at $x = 0$.

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

Integrating term by term, we conclude that $\int e^{-x^2} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C$.

Note. Since e^{-x^2} is an even function, its Taylor series only includes the terms x^{2n} (which are even) and not terms of the form x^{2n+1} (which are odd). See also the Taylor series that we got for $\cos(x)$ (which is even).

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Example 169. Determine the Taylor series of $f(x) = x^3 + 3x^2 + 3x + 3$ at $x = -1$.

Solution. By definition, the Taylor series in question is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \ldots$$

Clearly $f(-1) = 2$ and we to compute the other values $f^{(n)}(-1)$ as follows:

- $f'(x) = 3x^2 + 6x + 3$ so that $f'(-1) = 0$.
- $f''(x) = 6x + 6$ so that $f''(-1) = 0$.
- $f''(x) = \frac{3}{8x^{5/2}}$ so that $f'''(1) = \frac{3}{8}$.
- $f'''(x) = 6$ so that $f''(-1) = 6$.
- We note that $f^{(4)}(x) = 0$ so that $f^{(n)}(-1) = 0$ for all $n \geqslant 4$.

The Taylor series therefore is

$$f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 = 2 + \frac{6}{3!}(x+1)^3 = 2 + (x+1)^3.$$

Comment. For a polynomial $f(x)$, the Taylor series is the same polynomial just expanded around a different point. In particular, the Taylor series only has terms up to the degree of $f(x)$.

Example 170. Using familiar simpler power series, find the Taylor series at $x = 0$ for the following:

(a) $\frac{4}{2 + 7x^3}$

(b) $\frac{2}{1 + 3x} + e^{7x}$

Solution.

(a) Using the geometric series, we have $\frac{4}{2 + 7x^3} = \frac{4}{2} \cdot \frac{1}{1 - \left(-\frac{7}{2}x^3\right)} = 2\sum_{n=0}^{\infty} \left(-\frac{7}{2}x^3\right)^n = 2\sum_{n=0}^{\infty} \left(-\frac{7}{2}\right)^n x^{3n}$.

(b) Using both the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and the exponential series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$\frac{2}{1+3x} + e^{7x} = 2 \cdot \frac{1}{1 - (-3x)} + e^{7x} = 2\sum_{n=0}^{\infty} (-3x)^n + \sum_{n=0}^{\infty} \frac{(7x)^n}{n!} = \sum_{n=0}^{\infty} \left[2 \cdot (-3)^n + \frac{7^n}{n!}\right] x^n.$$

Example 171. Find a power series (about $x=0$) for $\frac{1}{1+x^2}$.

Solution. We plug $-x^2$ for x in the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ to get
$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$. This is valid for $|-x^2|<1$ or, equivalently, $|x|<1$.
In particular, this power series has radius of convergence 1.

Example 172. Find a power series (about $x=0$) for $\arctan(x)$.

Solution. Recall that $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$.
In Example 171, we observed that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ and that this power series converges if $|x|<1$.

We now integrate both sides of $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ to find a power series for $\arctan(x)$.

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$$

Hence, $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$. Since $\arctan(0) = 0$, it follows that $C = 0$.

Example 173. What is the exact interval of convergence in the previous example?

Solution. Since the convergence radius is 1, we know that the series converges for $|x|<1$, and diverges if $|x|>1$.
We don't yet know whether the series converges for $x=\pm 1$.

- For $x=1$, we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

  This is an **alternating series** because the terms are alternately positive and negative. Due to the alternating series test, the series converges ($a_n = \frac{1}{2n+1}$ is positive, decreasing and converges to 0).

- For $x=-1$, we get the series $-\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which is -1 times what we get for $x=1$.

  In particular, this series converges as well.

Our conclusion is that the exact interval of convergence is $[-1,1]$.

Comment. The series for $x=\pm 1$ are not **absolutely convergent** because, if we sum instead the absolute values of the terms, then we get $\sum_{n=0}^{\infty} \frac{1}{2n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$, and we know that this series diverges, because it is "half" of the harmonic series. This means that the series for $x=\pm 1$ are only **conditionally convergent**.

Since $\arctan(1) = \frac{\pi}{4}$, we conclude that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.
Note that $\arctan(-1) = -\arctan(1) = -\frac{\pi}{4}$, which explains why we got the same series times -1.

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Polar coordinates

Our usual coordinates (x, y) used to describe points in the plane are Cartesian coordinates. Polar coordinates are an alternative way of describing points.

The **polar coordinates** (r, θ) represent the point $(x, y) = r(\cos\theta, \sin\theta)$.

This means $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Important comment. Often, θ is taken from $[0, 2\pi)$ (but $(-\pi, \pi]$ is another popular choice), and, usually, $r \geqslant 0$.

Example 174. Which point (in Cartesian coordinates) has polar coordinates $r = 2$, $\theta = \frac{\pi}{6}$?

Solution. $(x, y) = r(\cos\theta, \sin\theta) = 2(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}, 1)$

[Draw a right triangle with angle $\frac{\pi}{6} = 30°$ to find $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$.]

Note. The polar coordinates $r = 2$, $\theta = \frac{\pi}{6} + 2\pi$ correspond to the same point $(\sqrt{3}, 1)$. Polar coordinates are not quite unique.

Note. Sometimes, we permit negative r. For instance, the polar coordinates $r = -2$, $\theta = \frac{\pi}{6} + \pi$ also describe the point $(\sqrt{3}, 1)$.

How to calculate the polar coordinates (r, θ) for (x, y)?

By Pythagoras, $r = \sqrt{x^2 + y^2}$, and the angle is $\theta = \operatorname{atan2}(y, x) \in (-\pi, \pi]$.

Why? It follows from $x = r\cos(\theta)$ and $y = r\sin(\theta)$ that $\frac{y}{x} = \tan(\theta)$. We therefore get $\theta = \arctan(\frac{y}{x})$ if θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (plot $\tan$ and $\arctan$ to remind yourself that $\arctan$ only takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$).

The function $\operatorname{atan2}$ is available in most programming languages (C, C++, PHP, Java, ...) and is a version of $\arctan(x)$ (or atan in those languages). Note that $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan(\theta)$. If our point is in the first or fourth quadrant, then $\theta = \arctan(\frac{y}{x}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Otherwise, $\theta = \arctan(\frac{y}{x}) + \pi$ (see next example).

Example 175. Find the polar coordinates, with $r \geqslant 0$ and $\theta \in [0, 2\pi)$ of $(5, 5)$ and $(-5, -5)$.

Solution. First, plot both points!

The polar coordinates of $(5, 5)$ are $r = 2\sqrt{5}$ and $\theta = \frac{\pi}{4}$.

The polar coordinates of $(-5, -5)$ are $r = 2\sqrt{5}$ and $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Note. $(5, 5)$ is in the first quadrant and $\theta = \arctan(\frac{y}{x}) = \arctan(1) = \frac{\pi}{4}$. On the other hand, $(-5, -5)$ is in the third quadrant, and so $\theta = \arctan(\frac{y}{x}) + \pi = \arctan(1) + \pi = \frac{5\pi}{4}$. [$\operatorname{atan2}$ allows us to avoid this distinction.]

Example 176. Describe a circle around the origin with radius 3 using Cartesian and polar coordinates.

Solution. Using Cartesian coordinates, the circle is described by $x^2 + y^2 = 3^2$.

Using polar coordinates, the circle is described by the even simpler equation $r = 3$.

Note. In this case, both coordinate equations are easy to see directly. We can, however, convert any equation in Cartesian coordinates to polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$. In our case, we would go from $x^2 + y^2 = 3^2$ to $(r\cos\theta)^2 + (r\sin\theta)^2 = 3^2$, which simplifies to $r^2 = 9$ or $r = 3$ (if we work with $r \geqslant 0$).

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Example 177. Convert the following equations to polar coordinates:

(a) $x + y = 3$

(b) $y = x^2 + 3x + 1$

Solution. We simply replace $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(a) $r\cos(\theta) + r\sin(\theta) = 3$

(b) $r\sin(\theta) = r^2\cos^2(\theta) + 3r\cos(\theta) + 1$

Example 178. Which shapes are described by the following equations?

(a) $r = 3$

(b) $\theta = \frac{\pi}{4}$

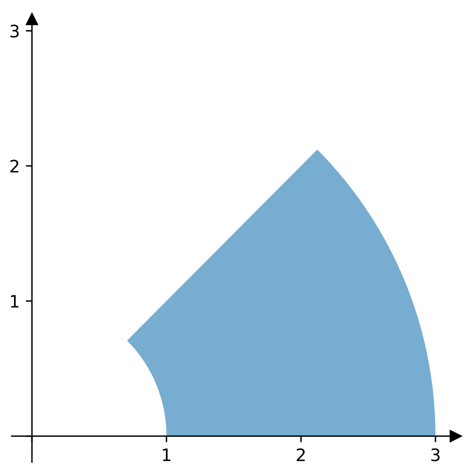
(c) $1 \leqslant r \leqslant 3$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$

Solution.

(a) This is a circle of radius 3 centered at the origin.

(b) This is the line through the origin that is angled $\frac{\pi}{4} = 45°$ up.
(In Cartesian coordinates, this is the line $y = x$.)

(c) The inequality $1 \leqslant r \leqslant 3$ describes an annulus (shaped like a CD: a disk with a hole).
The inequality $0 \leqslant \theta \leqslant \frac{\pi}{4}$ describes a cone.
Putting these two together, the region looks as follows:



Example 179. Describe the y-axis using polar coordinates.

Solution. $\theta = \pm\frac{\pi}{2}$ (just $\theta = \frac{\pi}{2}$ is enough if we also allow $r < 0$).

Alternatively. In Cartesian coordinates, the y-axis is described by the equation $x = 0$. In polar coordinates, this becomes $r\cos(\theta) = 0$. We can simplify this to $r = 0$ (that's just the origin) or $\cos(\theta) = 0$, where the latter becomes $\theta = \pm\frac{\pi}{2}$ (if we work with θ restricted to $(-y = x^2 + 3x + 1\pi, \pi]$).

Review. Polar coordinates

Parametric curves

Example 180. The unit circle is described by the Cartesian equation $x^2+y^2=1$ (in polar coordinates, the equation would be $r=1$). Instead of such coordinate equations, we can also describe the same curve by **parametrizing** it: $x=\cos(t)$, $y=\sin(t)$ with parameter $t\in[0,2\pi]$.

Comment. A curve can be parametrized in many ways. For instance, $x=t$, $y=\sqrt{1-t^2}$ with $t\in[-1,1]$ is another parametrization of the upper half-circle.

Remark. Note the difference in philosophies behind describing curves: an equation like $x^2+y^2=1$ is "exclusionary" because we start with all points (x,y) and then restrict to those with $x^2+y^2=1$. On the other hand, $x=\cos(t)$, $y=\sin(t)$ with $t\in[0,2\pi]$ is "inclusionary" because we are listing precisely the points on the curve.

We can work with parametric curves similarly to what we have been doing. For instance:

(arc length) The parametric curve $x=f(t)$, $y=g(t)$ with $t\in[a,b]$ has arc length

$$L=\int_a^b\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t=\int_a^b\sqrt{(f'(t))^2+(g'(t))^2}\,\mathrm{d}t.$$

Note that $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}$ equals $\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}$ from earlier.

Example 181. Using the parametric curve $x=r\cos(t)$, $y=r\sin(t)$ with parameter $t\in[0,2\pi]$, find the circumference of a circle of radius r.

Solution. $L=\int_0^{2\pi}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t=\int_0^{2\pi}\sqrt{r^2\sin^2(t)+r^2\cos^2(t)}\,\mathrm{d}t=\int_0^{2\pi}r\,\mathrm{d}t=2\pi r$

Given a parametric curve $x=f(t)$, $y=g(t)$, we can compute ordinary derivatives as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}\qquad\left[=\frac{g'(t)}{f'(t)}\right]$$

Likewise, writing $y'=\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}x}=\frac{\left(\frac{\mathrm{d}y'}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}\qquad\left[=\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{g'(t)}{f'(t)}\right)}{f'(t)}=\frac{g''(t)f'(t)-g'(t)f''(t)}{(f'(t))^3}\right]$$

Why? This is just the chain rule: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}$ (in our case, $\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{f'(t)}$)

It tells us that we can replace $\frac{\mathrm{d}}{\mathrm{d}x}$ (the derivative with respect to x) with $\frac{\mathrm{d}}{\mathrm{d}t}$ if we multiply the result with $\frac{\mathrm{d}t}{\mathrm{d}x}$.

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Example 182. Consider the parametric curve given by $x = t^2$, $y = t + 1$ with $t \geqslant 0$.

(a) Give an equivalent (non-parametric) Cartesian equation.

(b) Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point corresponding to $t = 1$.

Solution.

(a) If $x = t^2$, then $t = \sqrt{x}$, and so the curve is given by the Cartesian equation $y = \sqrt{x} + 1$.
Comment. In general, eliminating the parameter, as we did here, may be difficult or impossible.

(b) In order to see that we are really computing the same thing, we proceed both from the Cartesian equation $y = \sqrt{x} + 1$ as well as from the parametric equations:

- **(Cartesian equation)** Starting with $y(x) = \sqrt{x} + 1$, we have:

$$\begin{aligned} y'(x) &= \frac{1}{2\sqrt{x}} \quad \Longrightarrow \quad y'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} \\ y''(x) &= -\frac{1}{4}x^{-3/2} \quad \Longrightarrow \quad y''(1) = -\frac{1}{4} \end{aligned}$$

- **(parametric equations)** We now use $x = t^2$, $y = t + 1$ to compute the same quantities:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{1}{2t} \quad \Longrightarrow \quad \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{t=1} = \frac{1}{2} \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-1/(2t^2)}{2t} = -\frac{1}{4t^3} \quad \Longrightarrow \quad \left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]_{t=1} = -\frac{1}{4} \end{aligned}$$