Midterm #3 – Practice

MATH 126 — Calculus II
Midterm: Thursday, Nov 21, 2024

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. A nongraphing calculator (equivalent to the TI-30XIIS) is allowed on the exam (but not needed). No notes or further tools of any kind will be permitted on the midterm exam.

Problem 1. Go over all the quizzes since the last midterm exam!

To help you with that, there is a version of each quiz posted on our course website without solutions (of course, there are solutions, too).

Problem 2. Determine the Taylor polynomial of order 3 for $f(x)=\sqrt{x+2}$ at $x=2$.

Solution. By definition, the Taylor polynomial in question is given by

$$\sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3.$$

Clearly, $f(2)=\sqrt{4}=2$ and we to compute the other values $f^{(n)}(2)$ as follows:

- $f'(x)=\dfrac{1}{2\sqrt{x+2}}=\dfrac{1}{2}(x+2)^{-1/2}$ so that $f'(2)=\dfrac{1}{2\sqrt{4}}=\dfrac{1}{4}$.
- $f''(x)=-\dfrac{1}{4}(x+2)^{-3/2}$ so that $f''(2)=-\dfrac{1}{4\cdot 4^{3/2}}=-\dfrac{1}{4\cdot 8}=-\dfrac{1}{32}$.
- $f'''(x)=\left(-\dfrac{3}{2}\right)\cdot\left(-\dfrac{1}{4}\right)(x+2)^{-5/2}=\dfrac{3}{8}(x+2)^{-5/2}$ so that $f'''(2)=\dfrac{3}{8\cdot 4^{5/2}}=\dfrac{3}{8\cdot 32}=\dfrac{3}{256}$.

The Taylor polynomial therefore is

$$f(2)+f'(2)(x-2)+\frac{f''(2)}{2}(x-2)^2+\frac{f'''(2)}{6}(x-2)^3 = 2+\frac{1}{4}(x-2)-\frac{1}{64}(x-2)^2+\frac{1}{512}(x-2)^3.$$

Problem 3. Determine the following limits or state that the limit does not exist.

(a) $\displaystyle\lim_{n\to\infty}\frac{1+\log(n)}{1+n^{3/2}}$

(b) $\displaystyle\lim_{n\to\infty}\sin\left(\frac{5+3n^2}{\sqrt{1+2n^4}}\right)$

(c) $\displaystyle\lim_{n\to\infty}\frac{2-7n^2}{n(1+5n^2)}$

(d) $\displaystyle\lim_{n\to\infty}\frac{(5n^2+1)^2}{3n^4+2n^3+7}$

(e) $\displaystyle\lim_{n\to\infty}\frac{5^n+3^n}{4^n-1}$

(f) $\displaystyle\lim_{n\to\infty}\frac{3-2\log(n)}{4+2\log(n)}$

(g) $\displaystyle\lim_{n\to\infty}\frac{3n-2\log(n)}{4n+2\log(n)}$

(h) $\displaystyle\lim_{n\to\infty}\sqrt[n]{3n^2+1}$

(i) $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^n$

Solution.

(a) $\displaystyle\lim_{n\to\infty}\frac{1+\log(n)}{1+n^{3/2}}=0$ because $n^{3/2}$ grows much faster than $\log(n)$.

If we don't see this, we can apply L'Hospital to arrive at the same conclusion.

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(b) $\lim_{n\to\infty} \sin\left(\frac{5+3n^2}{\sqrt{1+2n^4}}\right) = \sin\left(\frac{3}{\sqrt{2}}\right)$ (because the dominant terms are $\frac{3n^2}{\sqrt{2n^4}} = \frac{3}{\sqrt{2}}$).

(c) $\lim_{n\to\infty} \frac{2-7n^2}{n(1+5n^2)} = 0$ (because the dominant terms are $\frac{-7n^2}{n\cdot 5n^2} = -\frac{7}{5n}$).

(d) $\lim_{n\to\infty} \frac{(5n^2+1)^2}{3n^4+2n^3+7} = \frac{5^2}{3} = \frac{25}{3}$ (because the dominant terms are $\frac{(5n^2)^2}{3n^4} = \frac{5^2}{3}$).

(e) $\lim_{n\to\infty} \frac{5^n+3^n}{4^n-1} = \infty$

You should be able to see this limit because 5^n grows more rapidly than all the other terms.

If we are unsure, we can divide top and bottom by 5^n (that doesn't change anything):

$$\lim_{n\to\infty} \frac{5^n+3^n}{4^n-1} = \lim_{n\to\infty} \frac{1+\left(\frac{3}{5}\right)^n}{\left(\frac{4}{5}\right)^n - \frac{1}{5^n}} = \infty$$

because $\left(\frac{3}{5}\right)^n$, $\left(\frac{4}{5}\right)^n$ and $\frac{1}{5^n}$ all approach 0 as $n\to\infty$.

(f) $\lim_{n\to\infty} \frac{3-2\log(n)}{4+2\log(n)} = \frac{-2}{2} = -1$ (because the dominant terms are $\frac{-2\log(n)}{+2\log(n)} = \frac{-2}{2}$).

(g) $\lim_{n\to\infty} \frac{3n-2\log(n)}{4n+2\log(n)} = \frac{3}{4}$ (because the dominant terms are $\frac{3n}{4n} = \frac{3}{4}$).

(h) $\lim_{n\to\infty} \sqrt[n]{3n^2+1} = \lim_{n\to\infty} \exp(\ln((3n^2+1)^{1/n})) = \lim_{n\to\infty} \exp\left(\frac{1}{n}\ln(3n^2+1)\right) = \exp(0) = 1$

At the end, we used that $\lim_{n\to\infty} \frac{1}{n}\ln(3n^2+1) = 0$. We can see this because the dominant terms are $\frac{1}{n}\ln(3n^2) = \frac{2}{n}\ln(3n)$ and we know that n grows more rapidly than the logarithm. Alternatively, we can use L'Hospital because the limit is in the indeterminate form "$\frac{\infty}{\infty}$": $\lim_{n\to\infty} \frac{\ln(3n^2+1)}{n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{\frac{1}{3n^2+1}\cdot 6n}{1} = \lim_{n\to\infty} \frac{6n}{3n^2+1} = 0$.

(i) $\lim_{n\to\infty} \left(1-\frac{1}{2n}\right)^n = \lim_{n\to\infty} \exp\left(\ln\left(\left(1-\frac{1}{2n}\right)^n\right)\right) = \lim_{n\to\infty} \exp\left(n\ln\left(1-\frac{1}{2n}\right)\right) = e^{-1/2}$

In the final step, we used that $\lim_{n\to\infty} n\ln\left(1-\frac{1}{2n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{2n}\right)}{\frac{1}{n}} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{2n}}\cdot\left(\frac{1}{2n^2}\right)}{-\frac{1}{n^2}} = -\frac{1}{2}$.

Problem 4. Determine whether the following series converge or diverge. In each case, indicate a reason.

(a) $\sum_{n=1}^{\infty} \frac{7\sqrt{n}+\log(n)}{n^2+4}$

(b) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{10\log(n)}$

(c) $\sum_{n=1}^{\infty} \frac{n-4\sqrt{n}}{n^2+\log(n)}$

(d) $\sum_{n=1}^{\infty} \frac{5n+2}{n^4+1}$

(e) $\sum_{n=2}^{\infty} \frac{1-3\sqrt{n}}{1+6\sqrt{n}}$

(f) $\sum_{n=0}^{\infty} \frac{(-4)^n}{7n^2+1}$

Solution.

(a) $\sum_{n=1}^{\infty} \frac{7\sqrt{n}+\log(n)}{n^2+4}$ converges by limit comparison with the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (because $p = \frac{3}{2} > 1$).

(b) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{10\log(n)}$ diverges because the terms $\frac{\sqrt{n}}{10\log(n)}$ do not converge to 0 as $n\to\infty$.

(c) $\sum_{n=1}^{\infty} \frac{n-4\sqrt{n}}{n^2+\log(n)}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series and diverges.

(d) $\sum_{n=1}^{\infty} \frac{5n+2}{n^4+1}$ converges by limit comparison with the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (because $p=3>1$).

(e) $\sum_{n=2}^{\infty} \frac{1-3\sqrt{n}}{1+6\sqrt{n}}$ diverges because $\frac{1-3\sqrt{n}}{1+6\sqrt{n}} \to -\frac{1}{2} \neq 0$ as $n \to \infty$.

(f) $\sum_{n=0}^{\infty} \frac{(-4)^n}{7n^2+1}$ diverges because the terms $\frac{(-4)^n}{7n^2+1}$ do not converge to 0 as $n \to \infty$.

Problem 5. Determine whether the following series converge or diverge. In each case, indicate a reason. If a series converges, further state whether it converges absolutely or only converges conditionally.

(a) $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+3}$

(d) $\sum_{n=0}^{\infty} \frac{n!}{4^n\sqrt{n^2+3}}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n\sqrt{n^2+3}}$

(f) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution.

(a) $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ diverges because $(-1)^n \sqrt{n} \nrightarrow 0$ as $n \to \infty$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ converges conditionally, because:

- $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ diverges by limit comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, and
- $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ converges by the alternating series test (since $\frac{1}{\sqrt{n^2+3}}$ is positive and decreases to zero).

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+3}$ converges absolutely because $\sum_{n=0}^{\infty} \frac{1}{n^2+3}$ converges (by limit comparison with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$).

(d) $\sum_{n=0}^{\infty} \frac{n!}{4^n\sqrt{n^2+3}}$ diverges because $\frac{n!}{4^n\sqrt{n^2+3}} \nrightarrow 0$ as $n \to \infty$.

(e) The series $\sum_{n=0}^{\infty} \frac{1}{4^n\sqrt{n^2+3}}$ converges by comparison with $\sum_{n=0}^{\infty} \frac{1}{4^n}$. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n\sqrt{n^2+3}}$ converges absolutely.

(f) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges absolutely. In fact, we know that $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

(If we don't recall this, we can quickly use the ratio test.)

Problem 6. Compute the following series (or state that it diverges):

(a) $\sum_{n=0}^{\infty} \frac{4^n - 6\cdot 2^n}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{4 - 2^n}{3^n}$ (c) $\sum_{n=1}^{\infty} \frac{4^n - 2}{3^n}$

Solution.

(a) $\sum_{n=0}^{\infty} \frac{4^n - 6\cdot 2^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n - 6\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{4}{5}} - 6\cdot\frac{1}{1-\frac{2}{5}} = 5 - 10 = -5$

(b) $\sum_{n=1}^{\infty} \frac{4 - 2^n}{3^n} = 4\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 4\left(\frac{1}{1-\frac{1}{3}} - \left(\frac{1}{3}\right)^0\right) - \left(\frac{1}{1-\frac{2}{3}} - \left(\frac{2}{3}\right)^0\right) = 4\left(\frac{3}{2} - 1\right) - (3-1) = 0$

(c) $\sum_{n=1}^{\infty} \frac{4^n - 2}{3^n}$ diverges because $\lim_{n\to\infty} \frac{4^n - 2}{3^n} = \infty$ (instead of 0).

Problem 7. For which values of x does $\sum_{n=2}^{\infty} \frac{(-1)^n x^n + 1}{3^n}$ converge? Compute the series for these values.

Solution.

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^n x^n + 1}{3^n} &= \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{3^n} + \sum_{n=2}^{\infty} \frac{1}{3^n} \\
&= \sum_{n=2}^{\infty} \left(-\frac{x}{3}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n \\
&= \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n - 1 + \frac{x}{3} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n - 1 - \frac{1}{3} \\
[\text{if } |-x/3| < 1] \quad &= \frac{1}{1-\left(-\frac{x}{3}\right)} - \left(\left(-\frac{x}{3}\right)^0 + \left(-\frac{x}{3}\right)^1\right) + \frac{1}{1-\frac{1}{3}} - \left(1 + \frac{1}{3}\right) \\
&= \frac{1}{1+\frac{x}{3}} - 1 + \frac{x}{3} + \frac{3}{2} - 1 - \frac{1}{3} \\
&= \frac{3}{3+x} + \frac{x}{3} - \frac{5}{6}
\end{aligned}$$

In particular, the series converges provided that $|-x/3| < 1$, or, equivalently, $|x| < 3$.

Comment. You could have also used the ratio test to determine convergence. In this case, this is not necessary because the series is a combination of two geometric series.

Problem 8. Determine the radius of convergence of the following power series.

(a) $\sum_{n=2}^{\infty} \frac{n!(x+1)^n}{10^n}$ (b) $\sum_{n=1}^{\infty} \frac{(5x-2)^n}{n3^n}$ (c) $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$

Recall that $\binom{2n}{n} = \frac{(2n)!}{n!n!}$.

Solution.

(a) We apply the ratio test with $a_n = \frac{n!(x+1)^n}{10^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(x+1)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{n!(x+1)^n}\right| = \left|\frac{x+1}{10}\right| \frac{(n+1)!}{n!} = \left|\frac{x+1}{10}\right| (n+1) \to \infty \text{ as } n \to \infty \text{ (unless } x = -1)$$

The ratio test implies that $\sum_{n=2}^{\infty} \frac{n!(x+1)^n}{10^n}$ diverges if $x \neq -1$.

Therefore, the radius of convergence is 0.

(b) We apply the ratio test with $a_n = \frac{(5x-2)^n}{n3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(5x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(5x-2)^n}\right| = \left|\frac{5x-2}{3}\right| \frac{n}{n+1} \rightarrow \left|\frac{5x-2}{3}\right| \text{ as } n \rightarrow \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{(5x-2)^n}{n3^n}$ converges if $\left|\frac{5x-2}{3}\right| < 1$. To focus on x, we can rewrite this as $|5x-2| < 3$ or, equivalently, $\left|x - \frac{2}{5}\right| < \frac{3}{5}$.

In this latter form, we see that the radius of convergence is $\frac{3}{5}$.

(c) We apply the ratio test with $a_n = \frac{(2n)!}{n!n!} x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2n+2)! x^{n+1}}{(n+1)!(n+1)!} \frac{n!n!}{(2n)! x^n}\right| = |x| \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \rightarrow 4|x| \text{ as } n \rightarrow \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} x^n$ converges if $4|x| < 1$ or, equivalently, $|x| < \frac{1}{4}$.

As such, the radius of convergence is $\frac{1}{4}$.

Problem 9. Consider the power series $\sum_{n=1}^{\infty} \frac{n \cdot 3^n}{5^n} (x-2)^n$.

(a) Determine the radius of convergence R.

(b) What is the exact interval of convergence?

(c) Let $f(x) = \sum_{n=1}^{\infty} \frac{n \cdot 3^n}{5^n} (x-2)^n$ for x such that $|x-2| < R$. Write down a power series for $f'(x)$.

Solution.

(a) We apply the ratio test with $a_n = \frac{n \cdot 3^n}{5^n} (x-2)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1) \cdot 3^{n+1} (x-2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n \cdot 3^n (x-2)^n}\right| = \frac{3}{5}|x-2| \frac{n+1}{n} \rightarrow \frac{3}{5}|x-2| \text{ as } n \rightarrow \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{n \cdot 3^n}{5^n} (x-2)^n$ converges if $\frac{3}{5}|x-2| < 1$ or, equivalently, $|x-2| < \frac{5}{3}$.

Therefore, the radius of convergence is $\frac{5}{3}$.

(b) We already know that the series converges if $|x-2| < \frac{5}{3}$, that is, if $x \in \left(2 - \frac{5}{3}, 2 + \frac{5}{3}\right)$. We also know that the series diverges if $|x-2| > \frac{5}{3}$. What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x = 2 \pm \frac{5}{3}$:

- $x = 2 - \frac{5}{3}$: in that case, the series is $\sum_{n=1}^{\infty} \frac{n \cdot 3^n}{5^n} \left(-\frac{5}{3}\right)^n = \sum_{n=1}^{\infty} n \cdot (-1)^n$. This series diverges because the terms $n \cdot (-1)^n$ do not converge to 0 as $n \rightarrow \infty$.

- $x = 2 + \frac{5}{3}$: in that case, the series is $\sum_{n=1}^{\infty} \frac{n \cdot 3^n}{5^n} \left(\frac{5}{3}\right)^n = \sum_{n=1}^{\infty} n$. Again, this series diverges (the terms n do not converge to 0).

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In conclusion, the exact interval of convergence is $\left(2-\frac{5}{3}, 2+\frac{5}{3}\right) = \left(\frac{1}{3}, \frac{11}{3}\right)$.

(c) $f'(x) = \sum_{n=1}^{\infty} \frac{n \cdot 3^n}{5^n} n(x-2)^{n-1} = \sum_{n=1}^{\infty} \frac{n^2 \cdot 3^n}{5^n} (x-2)^{n-1}$

Optional: By replacing the index n with $n+1$, we can rewrite this as $f'(x) = \sum_{n=0}^{\infty} \frac{(n+1)^2 \cdot 3^{n+1}}{5^{n+1}} (x-2)^n$.

Problem 10. Consider the power series $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} (x-1)^n$.

(a) Determine the radius of convergence R.

(b) What is the exact interval of convergence?

(c) Let $f(x) = \sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} (x-1)^n$ for x such that $|x-1| < R$. Write down a power series for $f'(x)$.

Solution.

(a) We apply the ratio test with $a_n = \frac{3^n}{\sqrt{n}} (x-1)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{\sqrt{n+1}} (x-1)^{n+1} \cdot \frac{\sqrt{n}}{3^n (x-1)^n}\right| = 3|x-1| \sqrt{\frac{n}{n+1}} \rightarrow 3|x-1| \text{ as } n \rightarrow \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} (x-1)^n$ converges if $3|x-1| < 1$ or, equivalently, $|x-1| < \frac{1}{3}$.

In particular, the radius of convergence is $\frac{1}{3}$.

(b) We already know that the series converges if $|x-1| < \frac{1}{3}$, that is, if $x \in \left(1-\frac{1}{3}, 1+\frac{1}{3}\right) = \left(\frac{2}{3}, \frac{4}{3}\right)$. We also know that the series diverges if $|x-1| > \frac{1}{3}$. What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x = \frac{2}{3}$ and $x = \frac{4}{3}$:

- $x = \frac{4}{3}$: in that case, the series is $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. This is the p-series with $p = \frac{1}{2}$ which we know diverges (because $p = \frac{1}{2} \leqslant 1$).

- $x = \frac{2}{3}$: in that case, the series is $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} \left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This is the alternating p-series with $p = \frac{1}{2}$ which we know converges (because $p = \frac{1}{2} > 0$).

  (We can use the alternating series test if we don't recall this fact about alternating p-series.)

In conclusion, the exact interval of convergence is $\left[\frac{2}{3}, \frac{4}{3}\right)$.

(c) $f'(x) = \sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} n(x-1)^{n-1} = \sum_{n=1}^{\infty} \sqrt{n} \cdot 3^n (x-1)^{n-1}$

Optional: By replacing the index n with $n+1$, we can rewrite this as $f'(x) = \sum_{n=0}^{\infty} \sqrt{n+1} \cdot 3^{n+1} (x-1)^n$.

Problem 11. Using the integral test, determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}}$ converges.

Solution. By the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}}$ converges if and only if the integral $\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^{3/2}}$ converges.

First, however, we should verify that the integral test indeed applies: the function $\frac{1}{x(\log x)^{3/2}}$ is obviously positive and continuous for $x \geqslant 2$. It is also decreasing, because $x\,(\log x)^{3/2}$ clearly increases.

Upon substituting $u = \log x$, we find that

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^{3/2}} = \int_{\log(2)}^\infty \frac{\mathrm{d}u}{u^{3/2}} = \left[-\frac{2}{u^{1/2}} \right]_{\log(2)}^\infty$$

is finite because $\lim_{u \to \infty} \left(-\frac{2}{u^{1/2}} \right) = 0$. Therefore, the series $\sum_{n=2}^\infty \frac{1}{n\,(\log n)^{3/2}}$ converges.