Midterm #3

MATH 126 — Calculus II
Thursday, Nov 21, 2024

Please print your name:

No notes, graphing calculators or other tools are permitted. There are 34 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points) Compute the following series (or state that it diverges):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n - 2}{3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{6 - 2^n}{3^n}$

Solution.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{6^n - 2}{3^n}$ diverges because $\displaystyle\lim_{n\to\infty} \frac{6^n - 2}{3^n} = \infty$ (instead of 0).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{6 - 2^n}{3^n} = 6\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 6\left(\frac{1}{1-\frac{1}{3}} - \left(\frac{1}{3}\right)^0\right) - \left(\frac{1}{1-\frac{2}{3}} - \left(\frac{2}{3}\right)^0\right) = 6\left(\frac{3}{2} - 1\right) - (3-1) = 1$

Problem 2. (6 points) Determine the Taylor polynomial of order 3 for $f(x) = \sqrt{x}$ at $x = 1$.

Solution. By definition, the Taylor polynomial in question is given by

$$\sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3.$$

Clearly, $f(1) = 1$ and we to compute the other values $f^{(n)}(1)$ as follows:

- $f'(x) = \dfrac{1}{2\sqrt{x}}$ so that $f'(1) = \dfrac{1}{2}$.
- $f''(x) = -\dfrac{1}{4x^{3/2}}$ so that $f''(1) = -\dfrac{1}{4}$.
- $f''(x) = \dfrac{3}{8x^{5/2}}$ so that $f'''(1) = \dfrac{3}{8}$.

The Taylor polynomial therefore is

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

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Problem 3. (4 points) Determine the following limits or state that the limit does not exist.

(a) $\lim_{n\to\infty} \frac{5^n+3^n}{4^n-1} =$ ∞

(b) $\lim_{n\to\infty} \frac{7n^2-8n}{2n^2+3} =$ $\frac{7}{2}$

(c) $\lim_{n\to\infty} \sqrt{\frac{3+2n^2}{1+n+n^2}} =$ $\sqrt{2}$

(d) $\lim_{n\to\infty} \cos\left(\frac{n}{n^2+1}\right) =$ $\cos(0)=1$

Problem 4. (8 points) Determine whether the following series converge or diverge. Make sure to indicate a reason!

(a) $\sum_{n=2}^{\infty} \frac{1-\log(n)}{1+\log(n)}$

☐ series converges ☐ series diverges

Indicate a reason:

The series diverges because $\frac{1-\log(n)}{1+\log(n)} \to -1 \neq 0$ as $n\to\infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

☐ series converges ☐ series diverges

Indicate a reason:

The series converges by limit comparison with the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Indeed, if $a_n = \frac{n+1}{n^3+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Hence, $\sum_{n=1}^{\infty} a_n$ converges because $\sum_{n=1}^{\infty} b_n$ does.

(c) $\sum_{n=2}^{\infty} \frac{7^n}{n^2\, 4^n}$

☐ series converges ☐ series diverges

Indicate a reason:

The series diverges because $\frac{7^n}{n^2\, 4^n} \to \infty \neq 0$ as $n\to\infty$.

(d) $\sum_{n=2}^{\infty} \frac{n+\sqrt{n}+7}{3n^2+1}$

☐ series converges ☐ series diverges

Indicate a reason:

The series diverges by limit comparison with the diverging harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Indeed, if $a_n = \frac{n+\sqrt{n}+7}{3n^2+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3}$.

Hence, $\sum_{n=2}^{\infty} a_n$ diverges because $\sum_{n=2}^{\infty} b_n$ does.

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Problem 5. (10 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}\, 3^n}$.

(a) Determine the radius of convergence R.

(b) What is the exact interval of convergence?

(c) Let $f(x) = \sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}\, 3^n}$ for x such that $|x+1| < R$. Write down a power series for $f'(x)$.

Solution.

(a) We apply the ratio test with $a_n = \frac{(x+1)^n}{\sqrt{n}\, 3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^{n+1}}{\sqrt{n+1}\, 3^{n+1}} \frac{\sqrt{n}\, 3^n}{(x+1)^n}\right| = \left|\frac{x+1}{3}\right| \frac{\sqrt{n}}{\sqrt{n+1}} \to \left|\frac{x+1}{3}\right| \text{ as } n \to \infty$$

Hence, the ratio test implies that $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 3^n}$ converges if $\left|\frac{x+1}{3}\right| < 1$, that is, $|x+1| < 3$.

In particular, the radius of convergence is 3.

(b) We already know that the series converges if $|x+1| < 3$, that is, if $x \in (-1-3, -1+3) = (-4, 2)$. We also know that the series diverges if $|x+1| > 3$. What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x = -4$ and $x = 2$:

- $x = 2$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(2+1)^n}{\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. This is the p-series with $p = \frac{1}{2}$ which we know diverges (because $p = \frac{1}{2} \leqslant 1$).

- $x = -4$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(-4+1)^n}{\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This is the alternating p-series with $p = \frac{1}{2}$ which we know converges (because $p = \frac{1}{2} > 0$).

  (We can use the alternating series test if we don't recall this fact about alternating p-series.)

In conclusion, the exact interval of convergence is $[-4, 2)$.

(c) $f'(x) = \sum_{n=1}^{\infty} \frac{n(x+1)^{n-1}}{\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n}(x+1)^{n-1}}{3^n}$

Optional: By replacing the index n with $n+1$, we can rewrite this as $f'(x) = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}(x+1)^n}{3^{n+1}}$.

Problem 6. (Bonus!) What is the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$? [We don't have the tools to evaluate this series, but you might remember from class.]

Solution. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(extra scratch paper)

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