Preparing for the Final

MATH 227 — Calculus III
Monday, May 2

Please print your name:

Problem 1. Redo all practice problems for Midterm 1 and Midterm 2!

(The problems below will only cover the new material.)

Problem 2. Go through all the quizzes!

Problem 3. Consider the vector field $\boldsymbol{F} = \begin{bmatrix} x\sin(xy^2 - z) \\ \ln(x^2 - z^2) \\ xzy \end{bmatrix}$.

(a) Compute $\operatorname{div} \boldsymbol{F}$.

(b) Compute $\operatorname{curl} \boldsymbol{F}$.

(c) Which of the expressions $\operatorname{div}\operatorname{curl}\boldsymbol{F}$, $\operatorname{div}\operatorname{div}\boldsymbol{F}$, $\operatorname{curl}\operatorname{div}\boldsymbol{F}$, $\operatorname{curl}\operatorname{curl}\boldsymbol{F}$ are nonsense? Compute those that make sense. [Computational! Save second part for last.]

(d) Express the divergence and curl of a vector field $\boldsymbol{G}$ using the operator ∇.

Solution.

(a) $\operatorname{div} \boldsymbol{F} = (\sin(xy^2 - z) + x\cos(xy^2 - z)y^2) + 0 + xy = \sin(xy^2 - z) + x\cos(xy^2 - z)y^2 + xy$

(b) $\operatorname{curl} \boldsymbol{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} x\sin(xy^2 - z) \\ \ln(x^2 - z^2) \\ xzy \end{bmatrix} = \begin{bmatrix} xz - \frac{-2z}{x^2 - z^2} \\ -x\cos(xy^2 - z) - zy \\ \frac{2x}{x^2 - z^2} - x\cos(xy^2 - z)2yx \end{bmatrix} = \begin{bmatrix} xz + \frac{2z}{x^2 - z^2} \\ -x\cos(xy^2 - z) - zy \\ \frac{2x}{x^2 - z^2} - 2x^2y\cos(xy^2 - z) \end{bmatrix}$

(c) $\operatorname{div}\operatorname{div}\boldsymbol{F}$ and $\operatorname{curl}\operatorname{div}\boldsymbol{F}$ are nonsense. (Why?)

$\operatorname{div}\operatorname{curl}\boldsymbol{F} = 0$ (this always the case; can you show why?)

$$\operatorname{curl}\operatorname{curl}\boldsymbol{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} xz + \frac{2z}{x^2 - z^2} \\ -x\cos(xy^2 - z) - zy \\ \frac{2x}{x^2 - z^2} - 2x^2y\cos(xy^2 - z) \end{bmatrix}$$

$$= \begin{bmatrix} -2x^2\cos(xy^2 - z) + 4x^3y^2\sin(xy^2 - z) + x\sin(xy^2 - z) + y \\ x + \frac{2(x^2 - z^2) + 4z^2}{(x^2 - z^2)^2} - \frac{2(x^2 - z^2) - 4x^2}{(x^2 - z^2)^2} + 4xy\cos(xy^2 - z) - 2x^2y^3\sin(xy^2 - z) \\ -\cos(xy^2 - z) + xy^2\sin(xy^2 - z) \end{bmatrix}$$

$$= \begin{bmatrix} -2x^2\cos(xy^2 - z) + 4x^3y^2\sin(xy^2 - z) + x\sin(xy^2 - z) + y \\ x + \frac{4(x^2 + z^2)}{(x^2 - z^2)^2} + 4xy\cos(xy^2 - z) - 2x^2y^3\sin(xy^2 - z) \\ -\cos(xy^2 - z) + xy^2\sin(xy^2 - z) \end{bmatrix}$$

(d) $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$ and $\operatorname{curl} \boldsymbol{G} = \nabla \times \boldsymbol{G}$. □

Problem 4. Let C be the positively oriented boundary of the region defined by $x^2 + y^2 \leqslant 4$, $x + y \leqslant 2$. Spell out (i.e. express as ordinary integrals) the following line integrals:

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(a) $\oint_C f(x,y)\mathrm{d}s$,

(b) $\oint_C f(x,y)\mathrm{d}x$,

(c) $\oint_C f(x,y)\mathrm{d}y$,

(d) $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, with $\boldsymbol{F} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$.

Solution. Sketch the region! The boundary breaks into two natural pieces: the line segment from $(2,0)$ to $(0,2)$, and the three quarters of the circle with radius 2 around the origin from $(0,2)$ back to $(2,0)$. We parametrize these pieces as

- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-t \\ t \end{bmatrix}$, from $t=0$ to $t=2$,
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}$, from $t=\pi/2$ to $t=2\pi$.

(a) $\oint_C f(x,y)\mathrm{d}s = \int_0^2 f(2-t,t)\cdot\sqrt{(-1)^2+1^2}\mathrm{d}t + \int_{\pi/2}^{2\pi} f(2\cos(t),2\sin(t))\cdot\sqrt{(-2\sin(t))^2+(2\cos(t))^2}\mathrm{d}t$
$= \sqrt{2}\int_0^2 f(2-t,t)\mathrm{d}t + 2\int_{\pi/2}^{2\pi} f(2\cos(t),2\sin(t))\mathrm{d}t$

(b) $\oint_C f(x,y)\mathrm{d}x = \int_0^2 -f(2-t,t)\mathrm{d}t + \int_{\pi/2}^{2\pi} -2\sin(t)f(2\cos(t),2\sin(t))\mathrm{d}t$

(c) $\oint_C f(x,y)\mathrm{d}y = \int_0^2 f(2-t,t)\mathrm{d}t + \int_{\pi/2}^{2\pi} 2\cos(t)f(2\cos(t),2\sin(t))\mathrm{d}t$

(d) $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_C f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$
$= \int_0^2 [-f(2-t,t)+g(2-t,t)]\mathrm{d}t + \int_{\pi/2}^{2\pi} [-2\sin(t)f(2\cos(t),2\sin(t)) + 2\cos(t)g(2\cos(t),2\sin(t))]\mathrm{d}t$ □

Problem 5. Let C be the positively oriented boundary of the quadrilateral T with vertices $(0,0)$, $(1,2)$, $(1,3)$, $(0,4)$.

(a) Evaluate the line integral $\oint_C xy\mathrm{d}x - x^2\mathrm{d}y$ using Green's Theorem.

(b) Evaluate the line integral $\oint_C xy\mathrm{d}x - x^2\mathrm{d}y$ directly.

(c) Do you expect the value of the line integral to change if C was replaced by a different curve through the same four points?

(d) Evaluate the line integral $\oint_C 2xy\mathrm{d}x + x^2\mathrm{d}y$ in the most economical way.

(e) Do you expect the value of this second line integral to change if C was replaced by a different loop through the same four points?

(f) Evaluate the line integral $\int_L 2xy\mathrm{d}x + x^2\mathrm{d}y$ where L is the line segment from $(0,0)$ to $(1,2)$ followed by the line segment from $(1,2)$ to $(1,3)$.

Solution.

(a) Make a sketch! Note that the region R enclosed by C is bounded by $0 \leqslant x \leqslant 1$, $2x \leqslant y \leqslant 4-x$.

Hence, $\oint_C xy\,\mathrm{d}x - x^2\mathrm{d}y = \iint_R (-2x-x)\mathrm{d}y\mathrm{d}x = -3\int_0^1 \int_{2x}^{4-x} x\mathrm{d}y\mathrm{d}x = -3\int_0^1 \Big[xy\Big]_{y=2x}^{y=4-x}\mathrm{d}x$
$= -3\int_0^1 (4x-3x^2)\mathrm{d}x = -3\Big[2x^2 - x^3\Big]_0^1 = -3.$

(b) We parametrize the four sides as (of course, any other parametrization is just as fine, as long as we are careful about the orientation)

- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$, from $t=0$ to $t=1$,
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}$, from $t=2$ to $t=3$,
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-t \\ 3+t \end{bmatrix}$, from $t=0$ to $t=1$,
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$, from $t=4$ to $t=0$.

Then, $\oint_C xy\,\mathrm{d}x - x^2\mathrm{d}y$
$= \int_0^1 (2t^2\cdot 1 - t^2\cdot 2)\mathrm{d}t + \int_2^3 (t\cdot 0 - 1^2\cdot 1)\mathrm{d}t + \int_0^1 ((1-t)(3+t)\cdot(-1) - (1-t)^2\cdot 1)\mathrm{d}t + \int_4^0 (0\cdot 0 - 0^2\cdot 1)\mathrm{d}t$
$= 0 - 1 + \int_0^1 (4t-4)\mathrm{d}t + 0 = -1 + \Big[2t^2 - 4t\Big]_0^1 = -3.$

(c) Yes. $\boldsymbol{F} = \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} xy \\ -x^2 \end{bmatrix}$ is not a conservative vector field because $M_y = x \neq N_x = -2x$. Hence, the line integral $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \oint_C xy\,\mathrm{d}x - x^2\mathrm{d}y$ depends on the precise path C.

(d) Note that $\boldsymbol{F} = \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$ is a conservative vector field because $M_y = 2x = N_x$. Hence, the line integral $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \oint_C 2xy\,\mathrm{d}x + x^2\mathrm{d}y = 0$ is zero (by the Fundamental Theorem) because C is a loop.

(e) No. Because the integrand $\boldsymbol{F} = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$ is a conservative vector field, the line integral $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \oint_C 2xy\,\mathrm{d}x + x^2\mathrm{d}y = 0$ is zero (by the Fundamental Theorem) for any loop C.

(f) Since the integrand is conservative with potential function $f(x,y) = x^2y$, we have $\int_L 2xy\,\mathrm{d}x + x^2\mathrm{d}y = f(1,3) - f(0,0) = 3$. (The precise path of L does not matter.) □

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Problem 6.

(a) Is the vector field $\boldsymbol{F} = \begin{bmatrix} 2\cos(2x-y) \\ \cos(2x-y)-1 \end{bmatrix}$ conservative? If so, determine a potential function.

(b) Is the vector field $\boldsymbol{F} = \begin{bmatrix} 2\cos(2x-y) \\ -\cos(2x-y)-1 \end{bmatrix}$ conservative? If so, determine a potential function.

(c) Is the vector field $\boldsymbol{F} = \begin{bmatrix} 3x^2y^2z - z \\ 2x^3yz - 3y^2 + z \\ x^3y^2 - x + y + 2 \end{bmatrix}$ conservative? If so, determine a potential function.

Solution.

(a) No. $\boldsymbol{F} = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ fails the Component Test since $M_y \neq N_x$.

(b) Yes. $\boldsymbol{F} = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ passes the Component Test: $M_y = 2\sin(2x-y) = N_x$.

To find a potential function, we start with $f_x = 2\cos(2x-y)$ (or the other equation). Integrating with respect to x, we therefore find that $f = \sin(2x-y) + C(y)$. Comparing derivatives with respect to y, we have $f_y = -\cos(2x-y) + C_y(y) = -\cos(2x-y) - 1$, which simplifies to $C_y(y) = -1$. Hence, $C(y) = -y + D$.

In conclusion, $f(x,y) = \sin(2x-y) - y$ is a potential function.

(c) Yes. $\boldsymbol{F}$ passes the Component Test: $\operatorname{curl}\boldsymbol{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} 3x^2y^2z - z \\ 2x^3yz - 3y^2 + z \\ x^3y^2 - x + y + 2 \end{bmatrix} = \begin{bmatrix} (2x^3y+1) - (2x^3y+1) \\ (3x^2y^2-1) - (3x^2y^2-1) \\ 6x^2yz - 6x^2yz \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

To find a potential function, we start with $f_x = 3x^2y^2z - z$ (or one of the other three equations). This implies that $f = x^3y^2z - xz + C(y,z)$. Comparing derivatives with respect to y, we have $f_y = 2x^3yz + C_y(y, z) = 2x^3yz - 3y^2 + z$, which simplifies to $C_y(y,z) = -3y^2 + z$. Hence, $C(y,z) = -y^3 + yz + D(z)$. So far, we found that $f = x^3y^2z - xz - y^3 + yz + D(z)$. Finally, comparing derivatives with respect to z, $f_z = x^3y^2 - x + y + D_z(z) = x^3y^2 - x + y + 2$. Simplified, this is $D_z = 2$, so $D = 2z + E$.

In conclusion, $f(x,y,z) = x^3y^2z - xz - y^3 + yz + 2z$ is a potential function.

□

Problem 7. Let C be the straight-line segment from $(0,3,-1)$ to $(4,-1,3)$. Let D be the curve parametrized by $\boldsymbol{r}(t) = t^2\boldsymbol{i} + (3-2t)\boldsymbol{j} + (2t-1)\boldsymbol{k}$, from $t=0$ to $t=2$.

(a) Evaluate the line integrals $\int_C x(z+1)^2 \mathrm{d}x + y\mathrm{d}z$ and $\int_D x(z+1)^2 \mathrm{d}x + y\mathrm{d}z$.

(b) Evaluate the line integrals $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ and $\int_D \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ with $\boldsymbol{F} = \begin{bmatrix} 3x^2y^2z - z \\ 2x^3yz - 3y^2 + z \\ x^3y^2 - x + y + 2 \end{bmatrix}$ (as in the previous problem).

(c) Write down an integral for the length of the curve D. No need to compute the integral; its numerical value is 7.19.

(d) Determine the average value of $f(x,y,z) = y + 3z$ on D.

Solution.

(a) Note that these integrals can also be written as $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ and $\int_D \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ with $\boldsymbol{F} = \begin{bmatrix} x(z+1)^2 \\ 0 \\ y \end{bmatrix}$. This vector field, however, is not conservative and so we cannot use the Fundamental Theorem of line integrals.

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We parametrize C using $\boldsymbol{r}(t) = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$, from $t=0$ to $t=1$. Using that parametrization,

$$\int_C x(z+1)^2 \mathrm{d}x + y\mathrm{d}z = \int_0^1 4t(4t)^2 \cdot 4\mathrm{d}t + (3-4t)\cdot 4\mathrm{d}t = \int_0^1 (256t^3 - 16t + 12)\mathrm{d}t = \left[64t^4 - 8t^2 + 12t\right]_0^1 = 68.$$

On the other, hand using the given parametrization $\boldsymbol{r}(t) = t^2\boldsymbol{i} + (3-2t)\boldsymbol{j} + (2t-1)\boldsymbol{k}$, from $t=0$ to $t=2$, for D,

$$\int_D x(z+1)^2 \mathrm{d}x + y\mathrm{d}z = \int_0^2 t^2(2t)^2 \cdot (2t)\mathrm{d}t + (3-2t)\cdot 2\mathrm{d}t = \int_0^2 (8t^5 - 4t + 6)\mathrm{d}t = \left[\frac{4t^6}{3} - 2t^2 + 6t\right]_0^2 = \frac{268}{3} \approx 89.3.$$

(b) From the previous problem, we know that $\boldsymbol{F}$ is conservative with potential function $f(x,y,z) = x^3y^2z - xz - y^3 + yz + 2z$. Since C and D are both curves from $(0,3,-1)$ to $(4,-1,3)$, we have, by the Fundamental Theorem of Line Integrals,

$$\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_D \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = f(4,-1,3) - f(0,3,-1) = (192 - 12 + 1 - 3 + 6) - (0 - 0 - 27 - 3 - 2) = 216.$$

(c) $\text{length}(D) = \int_D \mathrm{d}s = \int_0^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\mathrm{d}t = \int_0^2 \sqrt{4t^2+8}\mathrm{d}t \approx 7.19$

(d) $\int_D f\mathrm{d}s = \int_0^2 f(t^2, 3-2t, 2t-1)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\mathrm{d}t = \int_0^2 ((3-2t) + 3(2t-1))\sqrt{4t^2+8}\mathrm{d}t$

$= 8\int_0^2 t\sqrt{t^2+2}\mathrm{d}t = 4\int_2^6 \sqrt{u}\mathrm{d}u = \left[\frac{4u^{3/2}}{3/2}\right]_2^6 = \frac{8}{3}(6^{3/2} - 2^{3/2}) \approx 31.65$

The average therefore is $\text{avg} \approx \frac{31.65}{\text{length}(D)} \approx \frac{31.65}{7.19} \approx 4.40.$ □

Problem 8. Let R be the region defined by $2x + 2y + z \leqslant 6$, $x \geqslant 1$, $y \geqslant 1$, $z \geqslant 0$.

(a) Determine the volume of R.

(b) Determine the average value of $f(x,y,z) = x$ on R.

Solution.

(a) $\text{vol} = \int_1^2 \int_1^{3-x} \int_0^{6-2x-2y} \mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_1^2 \int_1^{3-x} (6 - 2x - 2y)\mathrm{d}y\mathrm{d}x = \int_1^2 \left[6y - 2xy - y^2\right]_{y=1}^{y=3-x}\mathrm{d}x$

$= \int_1^2 (4 - 4x + x^2)\mathrm{d}x = \left[4x - 2x^2 + \frac{x^3}{3}\right]_{x=1}^{x=2} = \frac{1}{3}$

(b) $\int_1^2 \int_1^{3-x} \int_0^{6-2x-2y} x\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_1^2 \int_1^{3-x} x(6 - 2x - 2y)\mathrm{d}y\mathrm{d}x = \int_1^2 x\left[6y - 2xy - y^2\right]_{y=1}^{y=3-x}\mathrm{d}x$

$= \int_1^2 (4x - 4x^2 + x^3)\mathrm{d}x = \left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_{x=1}^{x=2} = \frac{5}{12}$

$\text{avg} = \frac{1}{\text{vol}}\int_1^2 \int_1^{3-x} \int_0^{6-2x-2y} x\mathrm{d}z\mathrm{d}y\mathrm{d}x = \frac{5/12}{1/3} = \frac{5}{4}$ □

Problem 9. Let R be the region defined by $1 \leqslant x^2 + y^2 + z^2 \leqslant 4$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

(a) Write down an integral for the volume of R using spherical coordinates. Then, compute it.

(b) Write down an integral for the volume of R using cylindrical coordinates. Then, compute it.

[The bounds are somewhat tricky!]

(c) Write down an integral for the volume of R using cartesian coordinates. [Similar comment.]

(d) Write down an (ordinary) integral for the average value of some function $f(x,y,z)$ on R.

[It is up to you to choose which coordinates you prefer to use.]

Solution. R is an eighth of a ball of radius 2, with an inner ball of radius 1 removed. Its volume therefore is

$$\mathrm{vol}(R) = \frac{1}{8}\left[\frac{4\pi 2^3}{3} - \frac{4\pi 1^3}{3}\right] = \frac{7\pi}{6}.$$

We will use that value to check that our computations involving multiple integrals are correct.

(a) $$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{2}\left(\int_1^2 \rho^2 d\rho\right)\left(\int_0^{\pi/2}\sin\phi \, d\phi\right) = \frac{\pi}{2}\left[\frac{\rho^3}{3}\right]_1^2\left[-\cos\phi\right]_0^{\pi/2} = \frac{\pi}{2}\cdot\frac{7}{3}\cdot 1 = \frac{7\pi}{6}$$

(b) We have some choices about the order in which we consider the cylindrical coordinates r, θ, z. Here are two natural choices: (in each case, note that the inequality $1 \leqslant x^2+y^2+z^2 \leqslant 4$ becomes $1 \leqslant r^2+z^2 \leqslant 4$)

- An integral $d\theta dr dz$ (or $dr d\theta dz$):

  The tricky part is that the range for r changes in nature, depending on whether $0 \leqslant z \leqslant 1$ (in that case, r ranges from the smaller sphere [where $r^2+z^2=1$, so $r=\sqrt{1-z^2}$] to the larger sphere [where $r^2+z^2=4$, so $r=\sqrt{4-z^2}$]) or whether $1 \leqslant z \leqslant 2$ (in that case, r ranges from 0 to the larger sphere). We split the integral accordingly:

  $$\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{4-z^2}} \int_0^{\pi/2} r \, d\theta dr dz + \int_1^2 \int_0^{\sqrt{4-z^2}} \int_0^{\pi/2} r \, d\theta dr dz$$
  $$= \frac{\pi}{2}\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{4-z^2}} r \, dr dz + \frac{\pi}{2}\int_1^2 \int_0^{\sqrt{4-z^2}} r \, dr dz = \frac{\pi}{2}\int_0^1 \left[\frac{r^2}{2}\right]_{r=\sqrt{1-z^2}}^{r=\sqrt{4-z^2}} dz + \frac{\pi}{2}\int_1^2 \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4-z^2}} dz$$
  $$= \frac{\pi}{4}\int_0^1 3 dz + \frac{\pi}{4}\int_1^2 (4-z^2) dz = \frac{3\pi}{4} + \frac{\pi}{4}\left[4z - \frac{z^3}{3}\right]_1^2 = \frac{3\pi}{4} + \frac{\pi}{4}\frac{5}{3} = \frac{7\pi}{6}$$

- An integral $dz dr d\theta$ (or $dz d\theta dr$):

  The tricky part is that the range for z changes in nature, depending on whether $0 \leqslant r \leqslant 1$ (in that case, z ranges from the smaller sphere to the larger sphere) or whether $1 \leqslant r \leqslant 2$ (in that case, z ranges from 0 to the larger sphere). We split the integral accordingly:

  $$\int_0^{\pi/2}\int_0^1 \int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} r \, dz dr d\theta + \int_0^{\pi/2}\int_1^2 \int_0^{\sqrt{4-r^2}} r \, dz dr d\theta$$
  $$= \frac{\pi}{2}\int_0^1 r\left(\sqrt{4-r^2} - \sqrt{1-r^2}\right) dr + \frac{\pi}{2}\int_1^2 r\sqrt{4-r^2} dr$$
  $$= \frac{\pi}{2}\left[-\frac{2}{2\cdot 3}(4-r^2)^{3/2} + \frac{2}{2\cdot 3}(1-r^2)^{3/2}\right]_0^1 + \frac{\pi}{2}\left[-\frac{2}{2\cdot 3}(4-r^2)^{3/2}\right]_1^2$$
  $$= \frac{\pi}{6}\left[-3^{3/2} + 4^{3/2} - 1^{3/2}\right] + \frac{\pi}{6}\left[3^{3/2}\right] = \frac{\pi}{6}(2^3 - 1) = \frac{7\pi}{6}$$

(c) Recall that R is an eighth of a ball of radius 2, with an inner ball of radius 1 removed.

Because of symmetry, it doesn't matter in which order we consider the variables. We will write down an integral with $dz dy dz$. The overall range for x is $0 \leqslant x \leqslant 2$. With x fixed, the range for y is $0 \leqslant \sqrt{4-x^2}$.

The tricky part is that the range for z changes in nature, depending on whether $x^2+y^2 \leqslant 1$ (in that case, z ranges from the smaller sphere [where $x^2+y^2+z^2=1$, so $z=\sqrt{1-x^2-y^2}$] to the larger sphere [where $x^2+y^2+z^2=4$, so $z=\sqrt{4-x^2-y^2}$]) or whether $x^2+y^2 \geqslant 1$ (in that case, z ranges from 0 to the larger sphere). We split the integral accordingly:

$$\int_0^2 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{4-x^2-y^2}} \mathrm{d}z\mathrm{d}y\mathrm{d}x + \int_0^2 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Comment. Alternatively, if we insisted to write down a single integral,

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{a(x,y)}^{\sqrt{4-x^2-y^2}} \mathrm{d}z\mathrm{d}y\mathrm{d}x \text{ with } a(x,y) = \begin{cases} 0, & \text{if } 1-x^2-y^2 \leqslant 0, \\ \sqrt{1-x^2-y^2}, & \text{otherwise.} \end{cases}$$

(d) We can write down an integral for the average using any of the choices of coordinates:

- spherical coordinates:

$$\text{avg} = \frac{6}{7\pi} \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \rho^2 \sin\phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

- cylindrical coordinates:

$$\text{avg} = \frac{6}{7\pi} \left[\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{4-z^2}} \int_0^{\pi/2} f(r\cos\theta, r\sin\theta, z) r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z + \int_1^2 \int_0^{\sqrt{4-z^2}} \int_0^{\pi/2} f(r\cos\theta, r\sin\theta, z) r \mathrm{d}\theta \mathrm{d}r \mathrm{d}z \right]$$

- cartesian coordinates:

$$\text{avg} = \frac{6}{7\pi} \left[\int_0^2 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{4-x^2-y^2}} f(x,y,z) \mathrm{d}z\mathrm{d}y\mathrm{d}x + \int_0^2 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x,y,z) \mathrm{d}z\mathrm{d}y\mathrm{d}x \right]$$

□

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