Example 8. Geometrically, what is represented by $\theta = \frac{\pi}{4}$, $r \geqslant 0$? By $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$, $r \geqslant 3$?

(Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

- Write down both sides of Euler's identity for $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$.
  In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects all five fundamental constants).
- Realize that the complex number $\cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
  These are precisely the points on the unit circle!
- How can we make sense of the $e^{i\theta}$ in Euler's identity? One way, is to recall Taylor series from Calculus 2!

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Example 9. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $(e^x)^n = e^{nx}$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

Or, try to use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$

Example 10. $\int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x =$

Comment. We can compare this integral to the one from last class: since $\sqrt{1 - x^2}$ is bigger than $1 - x^2$ in the interval $[0, 1]$, we conclude that $\int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x$ is larger than $\int_0^1 (1 - x^2)\mathrm{d}x = \frac{2}{3}$.

Solution. Geometrically, we are computing the area of a quarter of the unit circle (make sure you see that!).
Hence, $\int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x = \frac{\pi}{4}$. [And, indeed, $\frac{\pi}{4} > \frac{2}{3}$.]

Solution. We substitute $x = \cos(\theta)$. (It is very important that you can do substitutions like that!)
Note that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin(\theta)\mathrm{d}0$. Moreover, $x = 0$ corresponds to $\theta = \frac{\pi}{2}$, and $x = 1$ to $\theta = 0$. Hence:

$$\int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x = \int_{\pi/2}^0 \sqrt{1 - \cos^2(\theta)}\, (-\sin(\theta))\mathrm{d}\theta$$

(Note that the new boundaries are actually very natural if you think in terms of polar coordinates!)
For homework, simplify and evaluate the new integral!

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