Review. vectors, length

Example 20.

- What is the length of $\boldsymbol{v} = \langle 3, 0, -1 \rangle$?

  Solution. $|\boldsymbol{v}| = \sqrt{3^2 + 0 + (-1)^2} = \sqrt{10}$

- Let us write $\boldsymbol{v} = \sqrt{10} \cdot \frac{\langle 3, 0, -1 \rangle}{\sqrt{10}} = \underbrace{\sqrt{10}}_{\text{length}} \cdot \underbrace{\left\langle \frac{3}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}} \right\rangle}_{\text{direction}}$. [We are starting to scale vectors!]

  It is clear from our construction that $\left\langle \frac{3}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}} \right\rangle$ has length 1 (and, of course, we can check that). A vector of length 1 is called a **unit vector**.

  Note. This decomposition into length and direction makes particular sense if you think about physical applications: if $\boldsymbol{v}$ is a force, then $|\boldsymbol{v}|$ is the magnitude of the force (Newtons). Or, if $\boldsymbol{v}$ is velocity, then $|\boldsymbol{v}|$ is the speed (m/s).

The **standard unit vectors** (in three dimensions) are

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1, 0 \rangle, \quad \boldsymbol{k} = \langle 0, 0, 1 \rangle.$$

Clearly, these vectors have length 1. By the way, if you work in higher dimensions, the standard unit vectors are commonly denoted $\boldsymbol{e}_1 = \langle 1, 0, 0, \ldots \rangle$, $\boldsymbol{e}_2 = \langle 0, 1, 0, \ldots \rangle$, ...

Example 21. Find the component forms of $\boldsymbol{j} - \boldsymbol{k}$ and $3\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$.

Solution. $\boldsymbol{j} - \boldsymbol{k} = \langle 0, 1, -1 \rangle$ and $3\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k} = \langle 3, -1, 7 \rangle$

[Note that we started adding vectors: $\boldsymbol{j} - \boldsymbol{k} = \langle 0, 1, 0 \rangle - \langle 0, 0, 1 \rangle = \langle 0, 1, -1 \rangle$]

Vectors can be added and scaled.

And, from an abstract point of view, there is nothing else. The other operations on vectors (like dot product and cross product) are possible in our 2 or 3-dimensional space, but not for all kinds of vectors in general.

Example 22. Let $\boldsymbol{v} = \langle 1, 2, 3 \rangle$ and $\boldsymbol{w} = \langle 2, -1, 1 \rangle$. Compute $\boldsymbol{v} + \boldsymbol{w}$, $4\boldsymbol{w}$ and $2\boldsymbol{v} - \boldsymbol{w}$:

Solution. $\boldsymbol{v} + \boldsymbol{w} = \langle 1 + 2, 2 - 1, 3 + 1 \rangle = \langle 3, 1, 4 \rangle$, $4\boldsymbol{w} = \langle 4 \cdot 2, 4 \cdot (-1), 4 \cdot 1 \rangle = \langle 8, -4, 4 \rangle$ and $2\boldsymbol{v} - \boldsymbol{w} = \langle 2 \cdot 1 - 2, 2 \cdot 2 - (-1), 2 \cdot 3 - 1 \rangle = \langle 0, 5, 5 \rangle$

Example 23. Let $\boldsymbol{v} = \langle 2, 2 \rangle$ and $\boldsymbol{w} = \langle 0, -3 \rangle$. Then $\boldsymbol{v} + \boldsymbol{w} = \langle 2, -1 \rangle$.

What is the geometric interpretation?

First, sketch the two vectors in standard position (i.e. as arrows starting at the origin).

See Figure 11.12 and the discussion of the **parallelogram law** in the book!

What is the geometric interpretation of $4\boldsymbol{v}$?

Example 24. Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vectors. Consider them in standard position, and let $\boldsymbol{x}$ be the vector connecting the head of $\boldsymbol{v}$ to the head of $\boldsymbol{w}$. Express $\boldsymbol{x}$ in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Solution. See Figure 11.14(a) in the book: $\boldsymbol{x} = \boldsymbol{w} - \boldsymbol{v}$

[If you have trouble seeing this, observe that $\boldsymbol{v} + \boldsymbol{x} = \boldsymbol{w}$ as in our previous example. Then solve for $\boldsymbol{x}$.]

[Another way to see this is to note that $-\boldsymbol{v}$ is the same arrow as $\boldsymbol{v}$ but with the direction reversed. With that in mind, we find $\boldsymbol{x} = (-\boldsymbol{v}) + \boldsymbol{w}$, again.]

Armin Straub
straub@southalabama.edu