Review. dot product and its properties

Example 28. Compute the following:

(a) $\langle 2, -1, 1\rangle \cdot \langle 1, 0, 3\rangle$

(b) $(2\boldsymbol{j} - \boldsymbol{k}) \cdot (\boldsymbol{i} + 3\boldsymbol{k})$

Solution. $\langle 2, -1, 1\rangle \cdot \langle 1, 0, 3\rangle = 5$ and $(2\boldsymbol{j} - \boldsymbol{k}) \cdot (\boldsymbol{i} + 3\boldsymbol{k}) = -3$

Example 29. Write $|\boldsymbol{v} - \boldsymbol{w}|^2$ as a dot product, and multiply it out.

Solution. $|\boldsymbol{v} - \boldsymbol{w}|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} = |\boldsymbol{v}|^2 - 2\boldsymbol{v}\cdot\boldsymbol{w} + |\boldsymbol{w}|^2$

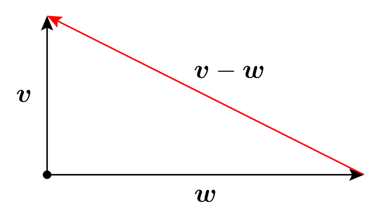
Comment. This is a vector version of $(x-y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v}\cdot\boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v}\times\boldsymbol{w}$, that we will soon see, we have $\boldsymbol{v}\times\boldsymbol{w} \neq \boldsymbol{w}\times\boldsymbol{v}$ (instead, $\boldsymbol{v}\times\boldsymbol{w} = -\boldsymbol{w}\times\boldsymbol{v}$).

Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **orthogonal** if and only if $\boldsymbol{v}\cdot\boldsymbol{w} = 0$.

Why? Short answer: Pythagoras!

Long answer: Consider two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in standard position, and consider the triangle as in the sketch. The angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is a right angle if and only if Pythagoras holds in this triangle:



$|\boldsymbol{v}|^2 + |\boldsymbol{w}|^2 = |\boldsymbol{v} - \boldsymbol{w}|^2$ (now use the previous example!)

$\iff |\boldsymbol{v}|^2 + |\boldsymbol{w}|^2 = |\boldsymbol{v}|^2 - 2\boldsymbol{v}\cdot\boldsymbol{w} + |\boldsymbol{w}|^2$ (next, cancel common terms)

$\iff 0 = -2\boldsymbol{v}\cdot\boldsymbol{w}$

$\iff \boldsymbol{v}\cdot\boldsymbol{w} = 0$ Which is what we wanted to show!

Replacing Pythagoras with the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$ holds in any triangle!), we obtain the following geometric interpretation of the dot product:

$\boldsymbol{v}\cdot\boldsymbol{w} = |\boldsymbol{v}|\,|\boldsymbol{w}|\cos\theta$ where $\theta \in [0, \pi]$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

What happens in the case $\boldsymbol{w} = \boldsymbol{v}$? Then, $\theta = 0$ and so...

Solving for θ, we obtain the following useful formula for the angle between two vectors:

The angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\theta = \arccos\left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{v}|\,|\boldsymbol{w}|}\right)$.

Example 30. What is the angle between $\boldsymbol{v} = \langle 1, 1\rangle$ and $\boldsymbol{w} = \langle 2, 0\rangle$?

Solution. Make a sketch! From the sketch it is obvious that the angle is $\theta = \frac{\pi}{4}$. Of course, this approach only worked because the vectors were chosen to be so pleasant.

Solution. $\theta = \arccos\left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{v}|\,|\boldsymbol{w}|}\right) = \arccos\left(\frac{2}{\sqrt{2}\,2}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ $[\boldsymbol{v}\cdot\boldsymbol{w} = 2,\ |\boldsymbol{v}| = \sqrt{2},\ |\boldsymbol{w}| = 2]$

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