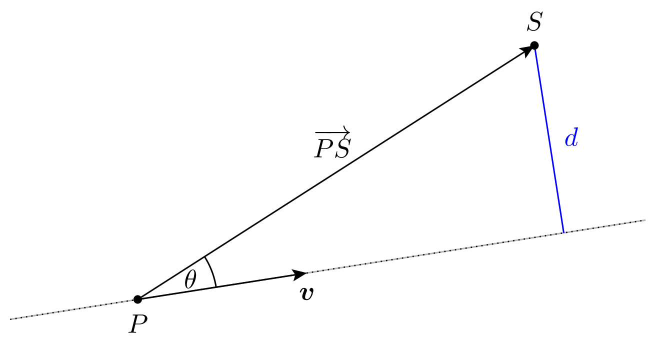
Example 44. (review) Find a parametrization for the line through $A = (1,1,1)$, $B = (2,1,3)$.

We pick the point $A = (1,1,1)$ and the direction $\overrightarrow{AB} = \langle 1,0,2\rangle$.
We then get the parametrization $P(t) = (1,1,1) + t\,\langle 1,0,2\rangle$, where $t \in (-\infty, \infty)$ is the parameter.
Note. We can spell out this parametrization component-wise as: $x = 1 + t$, $y = 1$, $z = 1 + 2t$.

> Consider the line through P, with direction $\boldsymbol{v}$.
>
> The **distance** between a point S and this line is $d = \dfrac{\left|\overrightarrow{PS} \times \boldsymbol{v}\right|}{|\boldsymbol{v}|}$.

Why? Recall that the distance d between a point S and a line is the length of the shortest path from S to the line.

- This shortest path is characterized by the fact that it is perpendicular to the line; see the sketch! [Compare our discussion on projections.]
- Because of that, $d = \left|\overrightarrow{PS}\right| \sin\theta$.
- Now, it only remains to recall that $\left|\overrightarrow{PS} \times \boldsymbol{v}\right| = \left|\overrightarrow{PS}\right| |\boldsymbol{v}| \sin\theta$.



Example 45. What is the distance between the point $S = (0,2,3)$ and the y-axis?

Solution. Make a sketch! In this case, it should be obvious that the distance is $d = 3$.

Solution. Our line is the y-axis. We can choose $P = (0,1,0)$ (or $P = (0,0,0)$, or any other point on the line) and $\boldsymbol{v} = \langle 0,7,0\rangle$ (or any multiple of this direction; we made this silly choice to emphasize that the final answer does not depend on these choices). Since, $\overrightarrow{PS} = \langle 0,1,3\rangle$, we find that the distance is

$$d = \frac{\left|\overrightarrow{PS} \times \boldsymbol{v}\right|}{|\boldsymbol{v}|} = \left|\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}\right| \Big/ \left|\begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}\right| = \left|\begin{bmatrix} -21 \\ 0 \\ 0 \end{bmatrix}\right| /7 = \frac{21}{7} = 3.$$

Example 46. What is the distance between $S = (2,3,0)$ and the line $P(t) = (1,1,1) + t\,\langle 1,0,2\rangle$?

Solution. We can choose $P = (1,1,1)$ and $\boldsymbol{v} = \langle 1,0,2\rangle$. Then, $\overrightarrow{PS} = \langle 1,2,-1\rangle$ and the distance is ... ... $= \sqrt{\frac{29}{5}}$.

Example 47. Parametrize the plane through $A = (1,1,1)$, $B = (2,1,3)$, $C(3,-1,1)$.

Solution. Now, we have 1 point & 2 directions: let's pick the point $A = (1,1,1)$ and $\overrightarrow{AB} = \langle 1,0,2\rangle$ as well as $\overrightarrow{AC} = \langle 2,-2,0\rangle$.
We then get the **parametrization** $P(s,t) = (1,1,1) + s\,\langle 1,0,2\rangle + t\,\langle 2,-2,0\rangle$.
Note. We now have two parameters s and t, which reflects the fact that planes are two-dimensional.
Note. Parametrizations of planes are very arbitrary! Not only can we replace the base point (here $(1,1,1)$) with any other point on the plane, but the two directions (here $\langle 1,0,2\rangle$ and $\langle 2,-2,0\rangle$) can be replaced with any two distinct directions parallel to the plane.
We will see next time, that the same plane can be much more economically represented by the equation $2x + 2y - z = 3$ (involving only 4 instead of 9 numbers!).

Armin Straub
straub@southalabama.edu