Example 48. Consider the plane through $(0,0,2)$ parallel to the xy-plane. Describe this plane by a parametrization as well as by an equation.

Solution. For a parametrization, we choose a point in the plane together with two directions parallel to the plane. For instance,

$$P(t) = \underbrace{(0,0,2)}_{\text{point}} + s\underbrace{\langle 1,0,0\rangle}_{\text{direction 1}} + t\underbrace{\langle 0,1,0\rangle}_{\text{direction 2}} = (s,t,2).$$

Note that we could have chosen a different point and different directions. For instance,

$$P(t) = \underbrace{(3,7,2)}_{\text{point}} + s\underbrace{\langle 2,2,0\rangle}_{\text{direction 1}} + t\underbrace{\langle 5,-1,0\rangle}_{\text{direction 2}} = (3+2s+5t, 7+2s-t, 2).$$

[In both cases, we get all points $(*,*,2)$, which are exactly the points on our plane.]

In this simple case, it is clear that the plane is described by the equation $z=2$ (or, $0x+0y+1z=2$ to emphasize what we are going to get in general). What is the geometric meaning of such equations?

- Each plane in 3D has a unique normal direction: that's the direction which is perpendicular to the plane. A **normal vector** is a vector in that direction. It is unique up to scaling.
- A plane is characterized by one point P_0 together with a normal vector $\boldsymbol{n}$; see Figure 11.39 in the book!
- Indeed, a point $P=(x,y,z)$ is on the plane $\Longleftrightarrow$ $\overrightarrow{P_0P}$ is perpendicular to $\boldsymbol{n}$ $\Longleftrightarrow$ $\overrightarrow{P_0P}\cdot\boldsymbol{n}=0$.
- In our case, we can choose $P_0=(0,0,2)$ and $\boldsymbol{n}=\langle 0,0,1\rangle$.

  Then, $\overrightarrow{P_0P}\cdot\boldsymbol{n}=0$ is $\begin{bmatrix} x-0 \\ y-0 \\ z-2 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}=0$. Simplified, we get $z-2=0$, or $z=2$, the desired equation.

Example 49. Find an equation for the plane through $A=(1,1,1)$, $B=(2,1,3)$, $C(3,-1,1)$.

Solution. $\overrightarrow{AB}=\langle 1,0,2\rangle$ and $\overrightarrow{AC}=\langle 2,-2,0\rangle$ are parallel to the plane.

Hence, we find a normal vector by computing $\boldsymbol{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\times\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}=\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$.

As point, we can choose $P_0=A=(1,1,1)$. Then, a point $P=(x,y,z)$ is on the plane $\Longleftrightarrow$ $\overrightarrow{P_0P}\cdot\boldsymbol{n}=0$.

That is, $\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}=0$ or $4(x-1)+4(y-1)-2(z-1)=0$, which simplifies to $4x+4y-2z=6$. Done!

[This equation is unique up to scaling: we could rescale it to $2x+2y-z=3$, for instance.]

Example 50. Find a vector normal to the plane $x+2y-z=3$.

Solution. $\boldsymbol{n}=(1,2,-1)$ (just taken from the coefficients; go through the previous example to see why the normal vector will always show up for these coefficients)

Comment. So, we understand the LHS of $x+2y-z=3$. The 3 on the other side is a measure for the distance of the plane from the origin (if $\boldsymbol{n}$ was a unit vector, then this would indeed be a distance).

Example 51. Find an equation for the plane parallel to $x+2y-z=3$ through $(1,1,1)$.

Solution. Since the planes are parallel, they have the same normal direction.

Our plane can therefore also be written as $x+2y-z=d$ for some d.

To find d, we use that $(x,y,z)=(1,1,1)$ is a point on the plane: $1+2\cdot 1-1=d$, so $d=0$.

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