In 2D, two lines are either parallel or they intersect in a unique point. In 3D (or higher dimensions), there is a new possibility: two lines can be **skew** (not intersecting but not parallel).

[Actually, two "random" lines should be skew. Think about it!]

Example 58. (intersecting two lines) Do the lines L_1 with parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 3t \\ 2+t \end{bmatrix}$ and L_2 with parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+t \\ -3-t \\ 1+2t \end{bmatrix}$ intersect? If so, find the point of intersection.

Note. This coming out of nowhere: the point on L_1 corresponding to $t=-1$ is $(2,-3,1)$. This same point is obtained from $t=0$ in the parametrization of L_2. So, this is the intersection we are looking for!

Solution. We have to solve the equations $\begin{array}{rcl} 1-t_1 &=& 2+t_2 \\ 3t_1 &=& -3-t_2 \\ 2+t_1 &=& 1+2t_2 \end{array}$. [Why did we use t_1 and t_2?! Look at the note!]

We can solve for t_1 and t_2 using the first two equations: $3\cdot \mathrm{eq}_1 + \mathrm{eq}_2$ is $3 = 3 + 2t_2$, which gives $t_2 = 0$. Substituting in eq_1, we get $1 - t_1 = 2 + 0$, which produces $t_1 = -1$.

We have to check whether eq_3 holds, too: $2 + (-1) \overset{??}{=} 1 + 2\cdot 0$. It does, and so we have found an intersection. The point of intersection is $(2,-3,1)$, corresponding to $t=-1$ in L_1 or $t=0$ in L_2.

Example 59. (intersecting two lines) Do the lines L_1 with parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 3t \\ 2+t \end{bmatrix}$ and L_2 with parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+t \\ 3+t \\ 1+2t \end{bmatrix}$ intersect? If so, find the point of intersection.

Solution. We have to solve the equations $\begin{array}{rcl} 1-t_1 &=& 2+t_2 \\ 3t_1 &=& 3+t_2 \\ 2+t_1 &=& 1+2t_2 \end{array}$.

$3\cdot \mathrm{eq}_1 + \mathrm{eq}_2$ is $3 = 9 + 4t_2$, which gives $t_2 = -\frac{3}{2}$. Substituting in eq_1, we get $1 - t_1 = 2 - \frac{3}{2}$ and so $t_1 = \frac{1}{2}$.

We have to check whether eq_3 holds, too: $2 + \frac{1}{2} \overset{??}{=} 1 + 2\left(-\frac{3}{2}\right)$. This is not true. This means that the lines do not intersect. They are skew.

Recall from Lecture 13:

The **distance** between a point S and the line through P, with direction $\boldsymbol{v}$ is $d = \frac{\left\lvert \overrightarrow{PS} \times \boldsymbol{v} \right\rvert}{\lvert \boldsymbol{v} \rvert}$.

Very similarly, we have:

The **distance** between a point S and the plane through P, with normal $\boldsymbol{n}$ is $d = \frac{\left\lvert \overrightarrow{PS} \cdot \boldsymbol{n} \right\rvert}{\lvert \boldsymbol{n} \rvert}$.

Why? See Figure 11.41 in the book.

Note that the distance is the length of the projection of $\overrightarrow{PS}$ onto $\boldsymbol{n}$.

Example 60. What is the distance between $S = (2,3,0)$ and the plane $2x + 2y - z = 3$?

Solution. From the equation, we see that $\boldsymbol{n} = (2,2,-1)$ is a normal vector for our plane.

We can choose any point P on the plane; an easy one is $P = (0,0,-3)$. [The z-intercept; see Lecture 15.]

Then, $d = \frac{\left\lvert \overrightarrow{PS} \cdot \boldsymbol{n} \right\rvert}{\lvert \boldsymbol{n} \rvert} = \left\lvert \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\rvert \Big/ \left\lvert \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\rvert = \frac{\lvert 4+6-3 \rvert}{\sqrt{4+4+1}} = \frac{7}{3}$.

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