Example 61. Are the lines parametrized as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1+6t \\ 3t \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3+2t \\ 1+t \end{bmatrix}$ the same?

Solution. The direction vectors are $\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. They are multiples of each other ($\boldsymbol{v}_1 = 3\boldsymbol{v}_2$), which means that the two lines are parallel. To decide whether the lines are the same we can check whether they share a point such as $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ (the base point of line 1). Since $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3+2t \\ 1+t \end{bmatrix}$ for $t = -1$, this point is also on line 2. Hence, the two lines are the same. [We could have also checked if the lines have 2 points in common.]

Example 62. Find the point R on the plane $2x + 2y - z = 3$ closest to $S = (2, 3, 0)$.

Solution. Realize that the vector $\overrightarrow{RS}$ has to perpendicular to the plane. Hence, the line through R and S has direction $\boldsymbol{n}$. We can find the point on the plane closest to S by intersecting the plane $2x + 2y - z = 3$ with the line through S and direction $\boldsymbol{n} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. That line has parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = S + t\boldsymbol{n} = \begin{bmatrix} 2+2t \\ 3+2t \\ -t \end{bmatrix}$.

We substitute the coordinates of points on the line into the equation for our plane:
$2(2+2t) + 2(3+2t) - (-t) = 3$ (which simplifies to $10 + 9t = 3$).

Solving for t, we obtain $t = -\frac{7}{9}$. The corresponding point on the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 \\ 13 \\ 7 \end{bmatrix}$. This is R.

Exercise. Compute the distance between R and S. (It has to be $\frac{7}{3}$, the distance we computed last time.)

Vector functions

Vector functions in 3D are functions of the form $\boldsymbol{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$.

Example 63. We have already seen examples in the form of parametrizations of curves:

$$P(t) = \begin{bmatrix} 1-t \\ 2+3t \\ 7t \end{bmatrix} \text{ (line)}, \quad P(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \text{ (circle)}, \quad P(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix} \text{ (``spiral up'')}.$$

Many of the concepts we are familiar with from (scalar) functions, extend to vector functions componentwise. For instance, the **derivative** of $\boldsymbol{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$ is $\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}'(t) = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$.

$\boldsymbol{r}'(t_0)$ is the **tangent vector** at $t = t_0$ to the curve parametrized as $\boldsymbol{r}(t)$.

The **tangent line** at $t = t_0$ is the line through $\boldsymbol{r}(t_0)$ parallel to $\boldsymbol{r}'(t_0)$.

Example 64. If $P(t) = \begin{bmatrix} 1-t \\ 2+3t \\ 7t \end{bmatrix}$, then $P'(t) = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$. The tangent vector is the same for all values of t. Clearly, this is just the direction vector.

Example 65. Find (a parametrization of) the tangent line to the curve $P(t) = \begin{bmatrix} e^{t^2} \\ t^2 - 1 \\ t \end{bmatrix}$ at $t = 0$.

Solution. The tangent line goes through $P(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and has direction $P'(0) = \begin{bmatrix} 2te^{t^2} \\ 2t \\ 1 \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, it is parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ t \end{bmatrix}$.

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