Review. Recall that the **arc length** of the curve $P(t)$ for $t \in [a,b]$ is $\int_a^b |\boldsymbol{v}(t)| \mathrm{d}t$.

Don't omit the $\mathrm{d}t$, it is important!
[And not just for technical reasons like substitution. Just think about units: the speed $|\boldsymbol{v}(t)|$ could be measured in miles/h. You absolutely need the little time spans $\mathrm{d}t$, measured in h, to get a length.]

Example 72. Evaluate the following integrals:

(a) $\int \begin{bmatrix} 1-t \\ e^{2t} \end{bmatrix} \mathrm{d}t$

Solution. $\int \begin{bmatrix} 1-t \\ e^{2t} \end{bmatrix} \mathrm{d}t = \begin{bmatrix} t - \frac{1}{2}t^2 + c_1 \\ \frac{1}{2}e^{2t} + c_2 \end{bmatrix} = \begin{bmatrix} t - \frac{1}{2}t^2 \\ \frac{1}{2}e^{2t} \end{bmatrix} + \boldsymbol{c}$

[Note that we need to allow different constants in each component. Or, use a vector constant $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.]

(b) $\int_0^1 \begin{bmatrix} te^{3t} \\ te^{3t^2} \end{bmatrix} \mathrm{d}t$

Solution. We need to compute two integrals:

By integration by parts, we get $\int_0^1 te^{3t} \mathrm{d}t = \left[t\left(\frac{1}{3}e^{3t}\right) \right]_0^1 - \int_0^1 \frac{1}{3}e^{3t} \mathrm{d}t = \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3t} \right]_0^1 = \frac{2}{9}e^3 + \frac{1}{9}$.

Substituting $u = 3t^2$ so that $\mathrm{d}u = 6t\mathrm{d}t$, we get $\int_0^1 te^{3t^2} \mathrm{d}t = \frac{1}{6}\int_0^3 e^u \mathrm{d}u = \frac{1}{6}[e^3 - 1]$.

Taken together, $\int_0^1 \begin{bmatrix} te^{3t} \\ te^{3t^2} \end{bmatrix} \mathrm{d}t = \begin{bmatrix} \frac{2}{9}e^3 + \frac{1}{9} \\ \frac{1}{6}e^3 - \frac{1}{6} \end{bmatrix}$.

Comment. If you feel a little uncomfortable with substitution or integration by parts, please review these soon! We will use them rather casually later on.

Example 73. (A very simple initial value problem) Solve the differential equation $\boldsymbol{r}''(t) = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with initial conditions $\boldsymbol{r}(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\boldsymbol{r}'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Solution. In this simple case, we can just integrate to get $\boldsymbol{r}'(t) = \begin{bmatrix} 0 \\ 0 \\ -t \end{bmatrix} + \boldsymbol{c}$.

Using $\boldsymbol{r}'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we find that $\boldsymbol{r}'(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. That is, $\boldsymbol{r}'(t) = \begin{bmatrix} 1 \\ 0 \\ -t \end{bmatrix}$.

Integrating once more, we obtain $\boldsymbol{r}(t) = \begin{bmatrix} t \\ 0 \\ -\frac{1}{2}t^2 \end{bmatrix} + \boldsymbol{d}$.

Using $\boldsymbol{r}(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, we find that $\boldsymbol{r}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{d} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. That is, $\boldsymbol{r}(t) = \begin{bmatrix} t+2 \\ 2 \\ -\frac{1}{2}t^2 + 2 \end{bmatrix}$.

Comment. We can interpret this simple equation in physical terms: if $\boldsymbol{r}(t)$ describes the position of a particle, then $\boldsymbol{r}''(t)$ is its acceleration and the differential equation matches a particle falling down due to gravitation. If we also know where exactly the particle is at $t = 0$ and what its velocity is at that time, then we can deduce the position at any time. (As we did, above.)

Armin Straub
straub@southalabama.edu