Example 75. What is the domain of $f(x,y) = \sin(x^2y)$?

Is it bounded or unbounded? Is it open? Is it closed? What is its boundary?

Solution. The domain is all of $\mathbb{R}^2$. It is unbounded, open, closed, and has no boundary points.

Partial derivatives

The partial derivate of a multivariate function is an ordinary derivative, where we regard all but one variable as constant:

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y, z_0)\Big|_{y=y_0}$$

We write the partial derivative of f with respect to y as either $\frac{\partial f}{\partial y}$ or f_y.

Example 76. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(a) $f(x,y) = 2x^2 - xy - y^2$

Solution. $f_x = 4x - y$ and $f_y = -x - 2y$

(b) $f(x,y) = x\sin(xy)$

Solution. $f_x = \sin(xy) + xy\cos(xy)$ and $f_y = x^2\cos(xy)$

Example 77. In contrast to the total derivative $\frac{\mathrm{d}}{\mathrm{d}x}$, the partial derivative $\frac{\partial}{\partial x}$ assumes variables to be independent:

$$\frac{\partial}{\partial x} xy = y \quad \text{vs.} \quad \frac{\mathrm{d}}{\mathrm{d}x} xy = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$

[If there is no dependence of y on x, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and we get the same answer.]

[$\frac{\partial}{\partial x}$ needs to be applied to a function $f(x, y, \ldots)$ and it needs to be clear what exactly all the variables x, $y, \ldots$ are. To avoid any possible ambiguity, we therefore often prefer to write $f_x(x, y, \ldots)$ instead.]

Example 78. Find all second-order partial derivatives of $f(x,y) = 2x^2 - xy - y^2$.

Solution. We are asked to find f_{xx}, f_{xy}, f_{yx}, f_{yy}. Using $f_x = 4x - y$, we find $f_{xx} = 4$ and $f_{xy} = -1$. Likewise, using $f_y = -x - 2y$, we find $f_{yy} = -2$ and $f_{yx} = -1$.

Example 79. Calculate f_{xy} and f_{yx} for $f(x,y) = x\sin(xy)$.

Solution.

$f_{xy} = \frac{\partial}{\partial y}[\sin(xy) + xy\cos(xy)] = x\cos(xy) + x\cos(xy) - x^2y\sin(xy) = 2x\cos(xy) - x^2y\sin(xy)$

$f_{yx} = \frac{\partial}{\partial x}[x^2\cos(xy)] = 2x\cos(xy) - x^2y\sin(xy)$ The two are equal!

(Mixed derivatives) $f_{xy} = f_{yx}$ (at points around which all of $f, f_x, f_y, f_{xy}, f_{yx}$ are continuous)

In other words, when taking partial derivatives, the order in which we take derivatives does not matter.

Example 80. Calculate f_{xyx} for $f(x,y) = x^3y^5 + \sqrt{x^2+1}$.

Solution. It saves us energy to compute f_{yxx} instead (because the piece $\sqrt{x^2+1}$ does not depend on y and so disappears when computing f_y) and by the result above $f_{xyx} = f_{yxx}$.

$f_y = 5x^3y^4$, $f_{yx} = 15x^2y^4$, $f_{xyx} = f_{yxx} = 30xy^4$.

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