Example 81. Calculate f_x, f_y and f_{xyy} for $f(x,y) = \sin(x^2+y)$.

Solution. $f_x = 2x\cos(x^2+y)$, $f_y = \cos(x^2+y)$.
Since the order doesn't matter, we choose to take the y derivatives first and compute $f_{yyx} = f_{xyy}$ instead (it is ever so slightly easier here): $f_{yy} = -\sin(x^2+y)$, $f_{yyx} = -2x\cos(x^2+y) = f_{xyy}$.

Example 82. Show that $f(x,y) = e^{-2x}\sin(2y)$ is a solution to the equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

[This is the two-dimensional Laplace equation, a famous example of a **partial differential equation**.]

Solution. $f_x = -2e^{-2x}\sin(2y)$, $f_{xx} = 4e^{-2x}\sin(2y)$, $f_y = 2e^{-2x}\cos(2y)$, $f_{yy} = -4e^{-2x}\sin(2y)$.
Hence, clearly, $f_{xx} + f_{yy} = 4e^{-2x}\sin(2y) - 4e^{-2x}\sin(2y) = 0$.

Calculus I. The best linear approximation to $f(x)$ at x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

(provided that f is differentiable at x_0). The right-hand side is the **linearization** of $f(x)$.

Comment. In Calculus II, you have learned to construct better and better approximations. For instance, the best quadratic approximation is $f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$ and the best cubic one is $f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$. Continuing this process indefinitely leads to to the Taylor series of $f(x)$ at x_0.

The **linearization** of $f(x,y)$ at (x_0,y_0) is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

We will soon say that $f(x,\ y)$ is differentiable at $(x_0,\ y_0)$ if and only if this linearization is a "good" approximation around (x_0,y_0).

Example 83. Find the linearization of $f(x,y) = x^3y^2$ at $(2,1)$.

Solution. $f_x = 3x^2y^2$ and $f_y = 2x^3y$. In particular, $f_x(2,1) = 12$ and $f_y(2,1) = 16$. Also, $f(2,1) = 8$.
Hence, the linearization of $f(x,y) = x^3y^2$ at $(2,1)$ is $L(x,y) = 8 + 12(x-2) + 16(y-1)$.
Comment. The graph of the linearization is the surface defined by $z = 8 + 12(x-2) + 16(y-1)$. We recognize that this equation describes a plane! (You can rewrite it as $12x + 16y - z = -32$.) This plane is tangent to the graph of $f(x,y) = x^3y^2$ at $(2,1)$.
[Just like the linearization at x_0 of a single-variable function $f(x)$ is a line tangent to the graph of $f(x)$ at x_0.]

Example 84. Find the equation for the plane tangent to the graph of $f(x,y) = x^3y^2$ at $(2,1)$.

Solution. As explained by the comment above, this tangent plane is $z = 8 + 12(x-2) + 16(y-1)$ (or, simplified, $12x + 16y - z = -32$).