Example 85. Find the linearization of $f(x,y,z)=x^2+y^2+z^2$ at $(1,2,3)$.

Solution. $f(1,2,3)=14$, $f_x(1,2,3)=\left[2x\right]_{x=1}=2$, $f_y(1,2,3)=\left[2y\right]_{y=2}=4$, $f_z(1,2,3)=\left[2z\right]_{z=3}=6$

Hence, the desired linearization is $L(x,y,z)=14+2(x-1)+4(y-2)+6(z-3)$.

The linearization of $f(x,y)$ at (x_0,y_0) is a good approximation around (x_0,y_0) if and only if $f(x,y)$ is **differentiable** at (x_0,y_0). [We make this precise below.]

Calculus I. $f(x)$ is **differentiable** at x_0 if, for some A,

$$f(x)=f(x_0)+A(x-x_0)+\underbrace{\text{“small” error}}_{=\varepsilon(x-x_0)\text{ with }\varepsilon\to 0\text{ as }x\to x_0},$$

in which case the derivative is $f'(x_0)=A$. You surely remember $f'(x)\approx\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$.

Calculus III. Likewise, $f(x,y)$ is **differentiable** at (x_0,y_0) if, for some A and B,

$$f(x,y)=f(x_0,y_0)+A(x-x_0)+B(y-y_0)+\underbrace{\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)}_{\text{“small” error}}. \qquad \text{(DIFF)}$$

In that case, $A=f_x(x_0,y_0)$ and $B=f_y(x_0,y_0)$. [Again, “small” means that $\varepsilon_1,\varepsilon_2\to 0$ as both $x\to x_0$ and $y\to y_0$.]

Theorem 86. If f_x and f_y are continuous around (x_0,y_0), then f is differentiable at (x_0,y_0).

This is good news because, in this case, we can avoid working with the technical condition (DIFF) directly. [Note that it is not enough that the *partial* derivatives $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist if we want to conclude that f is differentiable (or even continuous!) at $(x_0,\ y_0)$. See Figure 13.20 in the book for a cautionary example, which we might discuss in more detail later.]

The chain rule

If we write the condition (DIFF) of differentiability as $\Delta f\approx f_x\Delta x+f_y\Delta y$, divide both sides by Δt and take the limit as $\Delta t\to 0$, we arrive at the following chain rule.

(Chain rule, Part I) In the situation $f(x,y)$ with $x=x(t)$, $y=y(t)$ depending on t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x(t),y(t))=f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t),$$

or, for short, $\boxed{\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}}$. [Contrast this with the chain rule for $f(x)$ with $x=x(t)$!]

Fine print: $f(x,y)$ needs to be differentiable, and $x(t),y(t)$ need to be differentiable.

Example 87. Let $w=2x^2y$ and $x=\cos(2t)$, $y=t^3$. Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ (in terms of t) in two ways:

(a) by expressing w in terms of t and differentiating directly,

(b) by using the chain rule.

Solution.

(a) $\frac{\mathrm{d}w}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left[2\cos^2(2t)t^3\right]=4\cos(2t)(-2\sin(2t))\cdot t^3+2\cos^2(2t)\cdot 3t^2=-8t^3\cos(2t)\sin(2t)+6t^2\cos^2(2t)$.

(b) $\frac{\mathrm{d}w}{\mathrm{d}t}=\frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}=4xy\cdot(-2\sin(2t))+2x^2\cdot 3t^2$

Since we are asked to write this in terms of t, $\frac{\mathrm{d}w}{\mathrm{d}t}=-8t^3\cos(2t)\sin(2t)+6t^2\cos^2(2t)$.

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