Review. In the case $f(x)$ with $x = x(t)$, we have $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$.

In the case $f(x, y)$ with $x = x(t)$, $y = y(t)$, we have $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$.

Example 88. Let $w = \frac{x}{y} - \frac{y}{z}$ and $x = t^3$, $y = \sin(2t)$, $z = 1 + t$. Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ in two ways:
(a) by expressing w in terms of t and differentiating directly,
(b) by using the chain rule.

Solution.

(a) $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^3}{\sin(2t)} - \frac{\sin(2t)}{1+t}\right] = \frac{3t^2\sin(2t) - 2t^3\cos(2t)}{\sin^2(2t)} - \frac{2\cos(2t)(1+t) - \sin(2t)}{(1+t)^2}$

(b) $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{3t^2}{y} + \left(-\frac{x}{y^2} - \frac{1}{z}\right)2\cos(2t) + \frac{y}{z^2}$

Depending on our objective, we can now substitute x, y, z with their expressions in t:

$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{3t^2}{\sin(2t)} - \left(\frac{t^3}{\sin^2(2t)} + \frac{1}{1+t}\right)2\cos(2t) + \frac{\sin(2t)}{(1+t)^2}$

Check that this matches, of course, exactly what we computed before!

(Chain rule, Part II) In the situation $f(x, y)$ with $x = x(s, t)$, $y = y(s, t)$, we have, for short,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Example 89. Let $w = \ln(x + y^2)$ and $x = st$, $y = se^t$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ in two ways:
(a) by expressing w in terms of s, t and differentiating directly,
(b) by using the chain rule.

Solution.

(a) $w = \ln(st + s^2e^{2t})$

$\frac{\partial w}{\partial s} = \frac{t + 2s\,e^{2t}}{st + s^2e^{2t}}$

$\frac{\partial w}{\partial t} = \frac{s + 2s^2\,e^{2t}}{st + s^2e^{2t}}$

(b) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{x + y^2}\cdot t + \frac{2y}{x + y^2}\cdot e^t$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{x + y^2}\cdot s + \frac{2y}{x + y^2}\cdot se^t$

Again, substitute $x = st$, $y = se^t$ in these expressions and check that we get the same as before.

Remark 90. Observe that, in the situation $f(x, y)$ with $x = x(s, t)$, $y = y(s, t)$, the chain rule can be written (using a dot products) as

$$\frac{\partial f}{\partial s} = \underbrace{\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix}}_{\text{gradient of } f} \cdot \begin{bmatrix}\frac{\partial x}{\partial s}\\ \frac{\partial y}{\partial s}\end{bmatrix}, \quad \text{and} \quad \frac{\partial f}{\partial t} = \underbrace{\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix}}_{\text{gradient of } f} \cdot \begin{bmatrix}\frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial t}\end{bmatrix}.$$

We will denote the **gradient** of f by ∇f.

This makes the general version of the chain rule particularly natural, and will be very important to us in understanding geometric questions.

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