Example 91. If $z=f(x,y)$ and $y=g(x)$, then what is $\frac{dz}{dx}$?

Solution. Using the chain rule with $t=x$, we get $\frac{dz}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=f_x+f_y g'$.

Spelled out, you just derived the formula $\frac{d}{dx}f(x,g(x))=f_x(x,g(x))+f_y(x,g(x))g'(x)$.

(Chain rule, General Version) In the situation $f(x_1, x_2, \ldots)$ with $x_1 = x_1(t_1, t_2, \ldots)$, $x_2=x_2(t_1,t_2,\ldots)$, we have, for short,

$$\frac{\partial f}{\partial t_i}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\ldots \quad \text{for each } i.$$

We can write the chain rule even more compactly (and it may be easier to remember that way).

The **gradient** of f is $\nabla f=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\ldots\right)$. Also, write $\boldsymbol{x}=(x_1,x_2,\ldots)$ so that $\frac{\partial \boldsymbol{x}}{\partial t_i}=\left(\frac{\partial x_1}{\partial t_i},\frac{\partial x_2}{\partial t_i},\ldots\right)$.

Then, $\boxed{\frac{\partial f}{\partial t_i}=\nabla f\cdot\frac{\partial \boldsymbol{x}}{\partial t_i}}$. It is visible that this is a direct analog of the single-variable chain rule $\frac{df}{dt}=\frac{df}{dx}\frac{dx}{dt}$.

Tangent planes and normal lines

"Bended" lines are **curves**, and "bended" planes are **surfaces**.

On the other hand, one of the reasons we discussed lines and planes so extensively, is that every curve and every surface locally looks just like a line or a plane.

- Consider the surface $f(x,y,z)=0$.

  We have seen such equations, for instance, for planes ($2x-y+z-3=0$) and spheres ($(x-1)^2+(y-2)^2+(z-3)^2-25=0$).

- Further, consider the curve $\boldsymbol{r}(t)=\begin{bmatrix} x(t)\\ y(t)\\ z(t)\end{bmatrix}$. [Recall that $\boldsymbol{r}'(t)$ indicates the direction of the curve!]

  We have seen such parametrizations, for instance, for lines, circles and a spiral.

- Suppose the curve lies on the surface: that means $f(x(t),y(t),z(t))=0$.
- Differentiating this equation, we get $\frac{df}{dt}=\nabla f\cdot\boldsymbol{r}'=\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{bmatrix}\cdot\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{bmatrix}=0$. That is:

The gradient ∇f is orthogonal to the surface $f(x,y,z)=0$!

That's because the equation $\nabla f\cdot\boldsymbol{r}'=0$ from above is saying that ∇f is orthogonal to the tangent vector of any curve on our surface. (Strictly speaking, when we say that ∇f is orthogonal to the surface $f(x,y,z)=0$, we mean that ∇f is orthogonal to the plane tangent to the surface at the point in question.)

Example 92. (plane) The surface $f(x,y,z)=2x-y+z-3=0$ is a plane. What is ∇f?

Solution. $\nabla f=(2,-1,1)$. We already know that this is a normal vector for the plane!

Example 93. (sphere) What is a normal vector for the sphere $x^2+y^2+z^2=1$ at $(0,0,1)$?

Solution. Geometrically (make a sketch!), it is obvious that $(0,0,1)$ (or any multiple, of course) is a normal vector at the "north pole" $(0,0,1)$.

On the other hand, if $f(x,y,z)=x^2+y^2+z^2$, then $\nabla f=(2x,2y,2z)$. So, at a point (x,y,z) on the sphere, the direction $(2x,2y,2z)$ is normal. In particular, at $(0,0,1)$, the direction $\nabla f\big|_{(0,01)}=(0,0,2)$ is normal.

Armin Straub
straub@southalabama.edu