Example 94. What is a normal vector for the sphere $(x-2)^2+(y+1)^2+z^2-4=0$ at its south pole?

Solution. (geometric intuition) Geometrically (make a sketch!), it is obvious that $(0,0,1)$ (or any multiple, of course) is a normal vector at the south pole. [$(0,0,-1)$ might actually be a more natural choice if we let the normal vector point "outwards" at each point.]

Solution. On the other hand, if $f(x,y,z)=(x-2)^2+(y+1)^2+z^2-4$, then $\nabla f=(2(x-2),2(y+1),2z)$. In particular, at the south pole $(2,-1,-2)$, the direction $\nabla f\Big|_{(2,-1,-2)}=(0,0,-4)$ is normal.

[Note that the south pole has coordinates $(2,-1,-2)$, because our sphere has center $(2,-1,0)$ and radius 2.]

Example 95. Find equations for the tangent plane and normal line for the surface $f(x,y,z)=xy-2yz+z^2=0$ at the point $P=(0,1,2)$.

Solution. (using a gradient) $\nabla f=(y,x-2z,-2y+2z)$ so that $\nabla f(0,1,2)=(1,-4,2)$.
Therefore, the tangent plane is of the form $x-4y+2z=d$ and, since $P=(0,1,2)$ is on that plane, we find $d=0-4+4=0$.
The normal line is parametrized as $\boldsymbol{r}(t)=(0,1,2)+t(1,-4,2)$.

Solution. (using a linearization) $\nabla f=(y,x-2z,-2y+2z)$ so that $\nabla f(0,1,2)=(1,-4,2)$.
The linearization of $f(x,y,z)$ is $L(x,y,z)=1(x-0)-4(y-1)+2(z-2)$.
[Note that we have already computed the partial derivatives $1,-4,2$ in the previous solution.]
Since f is differentiable, this is a good approximation, and the surface $f(x,y,z)=0$ is approximated by (the plane!) $L(x,y,z)=x-4(y-1)+2(z-2)=0$.
[If we prefer, we can simplify $x-4(y-1)+2(z-2)=x-4y+2z$. However, the former has the nice property of reflecting that we are working around the point $(0,1,2)$.]

Example 96. Find the plane tangent to the graph of the curve $f(x,y)=x(1+y^2)$ at $(2,1)$.

Important! The graph of the curve $f(x,y)$ is the same as the surface $z=f(x,y)$ (or, $f(x,y)-z=0$).

Solution. (using a gradient) The question is the same as the following: find the plane tangent to the surface $g(x,y,z)=x(1+y^2)-z=0$ at $(2,1,4)$. [$(2,1,4)=(2,1,f(2,1))$]
Do it!

Solution. (using a linearization) The linearization of $f(x,y)$ at $(2,1)$ is $L(x,y)=4+2(x-2)+4(y-1)$ (do it!). Hence, the graph $z=f(x,y)$ is approximated by $z=L(x,y)$, that is, $z=4+2(x-2)+4(y-1)$, which is an equation for our tangent plane.
Optionally, we simplify it to $2x+4y-z-4=0$.

Armin Straub
straub@southalabama.edu