Example 97. Find the line tangent to the curve of intersection of the surfaces $xyz=1$ and $x^2+2y^2+z^2=4$ at the point $(1,1,1)$.

Solution. If we work very close to the point $(1,1,1)$, then the two surfaces are very nearly two planes. The line tangent to the curve of intersection is therefore just the line of intersection of these two planes (and we have learned earlier how to intersect two planes).

The surface $f(x,y,z)=xyz-1=0$ at $(1,1,1)$ has normal vector $\nabla f\Big|_{(1,1,1)}=(yz,xz,xy)\Big|_{(1,1,1)}=(1,1,1)$.

The surface $g(x,y,z)=x^2+2y^2+z^2-4=0$ at $(1,1,1)$ has normal $\nabla g\Big|_{(1,1,1)}=(2x,4y,2z)\Big|_{(1,1,1)}=(2,4,2)$.

The line we are looking for lies in both of the corresponding tangent planes, and hence is orthogonal to both normal vectors. We can thus find its direction via the cross product:

$\begin{bmatrix}1\\1\\1\end{bmatrix}\times\begin{bmatrix}2\\4\\2\end{bmatrix}=\begin{bmatrix}2-4\\2-2\\4-2\end{bmatrix}=\begin{bmatrix}-2\\0\\2\end{bmatrix}$. The line therefore is $\begin{bmatrix}1\\1\\1\end{bmatrix}+t\begin{bmatrix}-2\\0\\2\end{bmatrix}$.

Directional derivatives

Motivation and goal. Recall that the gradient ∇f is orthogonal to the surface $f(x,y,z)=0$ (or the curve $f(x,y)=0$, if we are in 2D). We will see a second interpretation of the gradient: namely, ∇f is the direction in which f increases most rapidly.

- Sketch some level curves for $f(x,y)=x^2+y^2$. (The level curves are circles.)
- Observe that, at each point, ∇f is orthogonal to the level curve $f(x,y)=c$. (Why?!)
- Realize that f does not change at all on each level curve. Flipping this around, it is indeed reasonable to expect that, at a given point, f changes the most in the direction orthogonal to the level curve.

  In other words: $\boxed{\nabla f \text{ is the direction in which } f \text{ increases most rapidly.}}$

The directional derivative generalizes the partial derivatives, which are the directional derivatives in direction $\boldsymbol{i}$ and $\boldsymbol{j}$.

The **directional derivative** of $f(x,y,\ldots)$ at P in direction of the unit vector $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}}f\Big|_P=\nabla f\Big|_P\cdot\boldsymbol{u}.$$

We will justify this easy-to-compute formula next time, starting with a limit defintion of the directional derivative.

For now, notice that, in the case $\boldsymbol{u}=\boldsymbol{i}$ (respectively, $\boldsymbol{u}=\boldsymbol{j}$), the directional derivative (as it should be!) is just the partial derivative with respect to x (respectively, with respect to y).

Example 98. Find the derivative of $f(x,y)=e^{2x}\cos(y)$ at $(0,0)$ in direction $\boldsymbol{v}=\boldsymbol{i}-\boldsymbol{j}$.

Solution. $\nabla f\Big|_{(0,0)}=(2e^{2x}\cos(y),-e^{2x}\sin(y))\Big|_{(0,0)}=(2,0)$

We need to replace $\boldsymbol{v}$ with the unit vector $\boldsymbol{u}=\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}=\frac{1}{\sqrt{2}}(1,-1)$ in the same direction.

Hence, our directional derivative is $D_{\boldsymbol{u}}f\Big|_{(0,0)}=\begin{bmatrix}2\\0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}=\sqrt{2}$.