The **directional derivative** of $f(x, y, \ldots)$ at $P$ in direction of the unit vector $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f\Big|_P = \nabla f\Big|_P \cdot \boldsymbol{u}.$$

The notation for directional derivatives is not standard. For instance, it is also written as $\nabla_{\boldsymbol{u}} f$.

Let us make sense of the directional derivative by starting with a definition using a limit. In this limit form, it is clear that this is indeed the derivative of $f$ in direction of the vector $\boldsymbol{u}$:

- For a function $f(x, y)$ and unit vector $\boldsymbol{u}$, denote the directional derivative of $f$ at $P = (x_0, y_0)$ in direction $\boldsymbol{u}$ as [Recall limit definition of derivative from Calculus I.]

  $$D_{\boldsymbol{u}} f\Big|_P = \lim_{h \to 0} \frac{f(x_0 + h u_1, y_0 + h u_2) - f(x_0, y_0)}{h}.$$

  Can you see that the right-hand side is the ordinary derivative $g'(0)$ of $g(h) = f(x_0 + h u_1, y_0 + h u_2)$?
- Applying the chain rule with $f(x, y)$ and $x = x_0 + h u_1$, $y = y_0 + h u_2$, we therefore get

  $$D_{\boldsymbol{u}} f\Big|_P = \left[\frac{\mathrm{d}}{\mathrm{d}h} f(x_0 + h u_1, y_0 + h u_2)\right]_{h=0} = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2 = \nabla f\Big|_P \cdot \boldsymbol{u}.$$

**Example 99.** Find the derivative of $f(x, y) = e^{2x}\cos(y)$ at $(0, 0)$ in direction $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j}$.

**Solution.** $\nabla f\Big|_{(0,0)} = (2e^{2x}\cos(y), -e^{2x}\sin(y))\Big|_{(0,0)} = (2, 0)$

We need to replace $\boldsymbol{v}$ with the unit vector $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{\sqrt{2}}(1, -1)$ in the same direction.

Hence, our directional derivative is $D_{\boldsymbol{u}} f\Big|_{(0,0)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sqrt{2}$.

For which direction $\boldsymbol{u}$ is the directional derivative as large as possible?

Using our knowledge of the dot product, observe that $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = |\nabla f|\,|\boldsymbol{u}|\cos\theta = |\nabla f|\cos\theta$. [What is $\theta$?] This is as large as possible if $\theta = 0$. In other words, if $\boldsymbol{u}$ is in the direction of $\nabla f$. This gives us another important interpretation of $\nabla f$:

$\nabla f$ is the direction in which $f$ increases most rapidly (direction of steepest ascent).

Correspondingly, $-\nabla f$ is the direction in which $f$ decreases most rapidly (direction of steepest descent).

**Example 100.** Find the derivative of $f(x, y) = 3x^2 - xy$ at $(1, 2)$ in direction $(3, 1)$.

In which direction does $f(x, y)$ at $(1, 2)$ increase most rapidly? Then find the derivative of the function in this direction.

In which direction does $f(x, y)$ at $(1, 2)$ decrease most rapidly? Then find the derivative of the function in this direction.

**Solution.**

- $\nabla f\Big|_{(0,0)} = (6x - y, -x)\Big|_{(1,2)} = (4, -1)$

  We need to replace $\boldsymbol{v} = (3, 1)$ with the unit vector $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{\sqrt{10}}(3, 1)$ in the same direction.

  Hence, our directional derivative is $(D_{\boldsymbol{u}} f)_{(1,2)} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{11}{\sqrt{10}}$.
- $f(x, y)$ at $(1, 2)$ increases most rapidly in the direction $(4, -1)$ (or, using a unit vector, $\boldsymbol{m} = \frac{1}{\sqrt{17}}(4, -1)$).

  The derivative in that direction is $(D_{\boldsymbol{m}} f)_{(1,2)} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \sqrt{17}$. (Of course, $\sqrt{17} > \frac{11}{\sqrt{10}}$.)
- Likewise, $f(x, y)$ at $(1, 2)$ decreases most rapidly in the direction $-\boldsymbol{m}$ with $(D_{-\boldsymbol{m}} f)_{(1,2)} = -\sqrt{17}$.