Example 101. Check out Figure 13.25 in our book. It shows a real-world map with level curves. Why are rivers flowing in such a way that they are orthogonal to each level curve.

Extreme values

Calculus I. A local extremum of $f(x)$ at an interior point a of its domain can only occur at a **critical point** (points such that $f'(a)=0$, or at which f is not differentiable).

- If $f''(a)<0$, then f has a local maximum at a.
- If $f''(a)>0$, then f has a local minimum at a.
- Otherwise, we need to further investigate. [Consider, for instance, $f(x)=x^3$ and $f(x)=x^4$.]

(First derivative test) If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain, then $\nabla f\Big|_{(a,b)}=0$ (assuming the partial derivatives exist).

Calculus III. A local extremum of $f(x,y)$ at an interior point (a,b) of its domain can only occur at a **critical point** (points at which $\nabla f=0$, or at which f is not differentiable).

- If $f_{xx}f_{yy}-f_{xy}^2>0$: [$f_{xx}f_{yy}-f_{xy}^2=\det\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy}\end{bmatrix}$ is called the **Hessian**.]
  - If $f_{xx}<0$ (or, equivalently, $f_{yy}<0$) then f has a local maximum at (a,b).
  - If $f_{xx}>0$ (or, equivalently, $f_{yy}>0$) then f has a local minimum at (a,b).
- If $f_{xx}f_{yy}-f_{xy}^2<0$ then f has a saddle point at (a,b).
- Otherwise, we need to further investigate.

Comment (if you know some linear algebra). The condition for a local min is that all eigenvalues of the Hessian matrix are positive. In 2D, $f_{xx}f_{yy}-f_{xy}^2>0$ means that both eigenvalues have the same sign (since the determinant is the product of the eigenvalues). Then $f_{xx}>0$ (or $f_{yy}>0$) implies that both are positive.

Example 102. (very simple) Find the local extreme values of $f(x,y)=x^2+y^2$.

Solution. Obviously, the origin is the (global) minimum and there are no other local extrema. (This is clear from a sketch. You can also see it algebraically by noting that $f(0,0)=0$ and that $f(x,y)>0$ otherwise.)

Solution. Note that $\nabla f=\begin{bmatrix} 2x \\ 2y\end{bmatrix}$. To find the critical points, we need to solve the two equations $2x=0$ and $2y=0$. Clearly, the only solution is $x=0$ and $y=0$. So, the only critical point is $(0,0)$.

To see if $(0,0)$ is indeed a local extremum, we compute $f_{xx}f_{yy}-f_{xy}^2=2\cdot 2-0=4>0$ and $f_{xx}=2>0$ (or, $f_{yy}>0$). We conclude that $(0,0)$ is a local minimum.

(Since the domain is all of $\mathbb{R}^2$ and there are no other local extrema, it follows that $(0,0)$ is a global minimum.)

Example 103. Find the local extreme values (and saddles) of $f(x,y)=3y^2-2y^3-3x^2+6xy$.

Solution. To find the critical points, we need to solve the two equations $f_x=-6x+6y=0$ and $f_y=6y-6y^2+6x=0$ for the two unknowns x,y.

[A general strategy is to solve one equation for one variable (in terms of the other), and substitute that in the other equation. Then we have a single equation in a single variable, which we can solve.]

Here, the first equation simplifies to $x=y$. Substituting that in the second equation, we get $6y-6y^2+6y=12y-6y^2=6y(2-y)=0$. Hence, $y=0$ or $y=2$.

If $y=0$ then $x=y=0$, and we get the point $(0,0)$. If $y=2$ then $x=y=2$, and we get the point $(2,2)$.

In conclusion, the critical points are $(0,0)$, $(2,2)$.

$\left[f_{xx}f_{yy}-f_{xy}^2\right]_{(0,0)}=\left[(-6)\cdot(6-12y)-6^2\right]_{(0,0)}=-72<0$. Hence, $(0,0)$ is a saddle point.

$\left[f_{xx}f_{yy}-f_{xy}^2\right]_{(2,2)}=\left[(-6)\cdot(6-12y)-6^2\right]_{(2,2)}=72>0$ and $f_{xx}=-6<0$. Hence, $(2,2)$ is a local max.

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