Multiple integrals

Review. On Monday, Dr. Carter showed you how to interpret the volume below $z = f(x, y)$ and above a region R in the xy-plane as a double integral $\int \int_R f(x, y) \mathrm{d}A$.

Example 107. Consider the **iterated integral** $\int_0^2 \int_{-1}^1 (x - y)\, \mathrm{d}y \mathrm{d}x$.

(a) Evaluate the integral.
(b) Determine and sketch the region of integration.
(c) Interchange the order of integration.
(d) Evaluate that second integral. (Make sure the answer is the same!)

Solution.

(a) $\int_0^2 \int_{-1}^1 (x - y)\, \mathrm{d}y \mathrm{d}x = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{y=-1}^{y=1} \mathrm{d}x = \int_0^2 2x \mathrm{d}x = \left[x^2\right]_{x=0}^{x=2} = 4.$

(b) The integral informs us that the region of integration R is determined by $0 \leqslant x \leqslant 2$, $-1 \leqslant y \leqslant 1$.
This is a rectangle!
Important note. Another way to write our integral is $\int \int_R (x - y) \mathrm{d}A$ where R is the region above.
[Instead of $\mathrm{d}A$, it is common to still write $\mathrm{d}y\mathrm{d}x$ or $\mathrm{d}x\mathrm{d}y$. The A indicates area.]

(c) The region R is also described by $-1 \leqslant y \leqslant 1$, $0 \leqslant x \leqslant 2$ (we only changed the order of the equations, which doesn't do anything). From that point of view, it is clear that our integral also equals $\int_{-1}^1 \int_0^2 (x - y)\, \mathrm{d}x \mathrm{d}y$, where the order of integration is now interchanged.

(d) $\int_{-1}^1 \int_0^2 (x - y)\, \mathrm{d}x \mathrm{d}y = \int_{-1}^1 \left[\frac{x^2}{2} - yx\right]_{x=0}^{x=2} \mathrm{d}y = \int_{-1}^1 (2 - 2y) \mathrm{d}y = \left[2y - y^2\right]_{y=-1}^{y=1} = 1 - (-3) = 4$

Example 108. Consider the integral $\int_0^2 \int_{x^2}^{2x} (4x + 2)\, \mathrm{d}y \mathrm{d}x$.

(a) Evaluate the integral.
(b) Determine and sketch the region of integration.
(c) Interchange the order of integration.
(d) Evaluate that second integral. (Make sure the answer is the same!)

Solution.

(a) $\int_0^2 \int_{x^2}^{2x} (4x + 2)\, \mathrm{d}y \mathrm{d}x = \int_0^2 (4x + 2)(2x - x^2) \mathrm{d}x = \int_0^2 (4x + 6x^2 - 4x^3) \mathrm{d}x = \left[2x^2 + 2x^3 - x^4\right]_0^2 = 8$

(b) The integral informs us that the region of integration R is $0 \leqslant x \leqslant 2$, $x^2 \leqslant y \leqslant 2x$.
[For sketching, realize that the region is bounded by $x = 0$, $x = 2$, $y = x^2$, $y = 2x$.]
See Figure 14.16 in our book for the actual sketch.

(c) To interchange the order of integration, we observe that the possible range for y is $0 \leqslant y \leqslant 4$. Fix a value of y (this means taking a **horizontal cross-sections** of our region), and think about the corresponding range of x. This range is $y/2 \leqslant x \leqslant \sqrt{y}$ (from the point on $y = 2x$ [i.e. $x = y/2$] to the point on $y = x^2$ [i.e. $x = \sqrt{y}$]).
To summarize, the region $0 \leqslant x \leqslant 2$, $x^2 \leqslant y \leqslant 2x$ can also be described by $0 \leqslant y \leqslant 4$, $y/2 \leqslant x \leqslant \sqrt{y}$.
Hence, interchanging the order of integration produced $\int_0^2 \int_{x^2}^{2x} (4x + 2) \mathrm{d}y \mathrm{d}x = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2) \mathrm{d}x \mathrm{d}y$.
Note. Taking **vertical cross-sections** of our region leads us back to the integral we started with.

(d) $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2) \mathrm{d}x \mathrm{d}y = \ldots = \int_0^4 \left(y + 2\sqrt{y} - \frac{y^2}{2}\right) \mathrm{d}y = \ldots = 8$ [Is it clear to you how to fill the blanks?]

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