You might not care so much about areas and volumes... Here is one reason why, if you care about any function, you would still care about the integrals we are currently thinking about.

- $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ is the **average value** of $f(x)$ on the (1D) region $[a,b]$.
- $\frac{1}{\text{area}(R)}\iint_R f(x,y)\mathrm{d}y\mathrm{d}x$ is the **average value** of $f(x,y)$ on the (2D) region R.
- $\frac{1}{\text{vol}(D)}\iiint_D f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$ is the **average value** of $f(x,y,z)$ on the (3D) region D.

Why? Recall how you would calculate the average of, say, the invidual (function) values $1,2,5$. The average value is $\frac{1}{3}(1+2+5)$. First, we add (or integrate) all values. Then, we divide by what we get if we had likewise just added (or integrated) 1's. (One way to think about this is that the average has to come out as 1 if the (function) values are all 1.)

Cartesian, cylindrical and spherical coordinates in 3D

Example 119. Write down an integral for the volume of the ball $x^2+y^2+z^2 \leqslant R^2$ (using cartesian coordinates).

Solution. $\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \mathrm{d}z\mathrm{d}y\mathrm{d}x$

The procedure (details omitted here) is as in yesterday's example. Make sure you understand and can do it!

It is not fun to try and compute this integral. We will see that cylindrical coordinates or spherical coordinates make this task much simpler.

(Cylindrical coordinates) $(x,y,z) \longleftrightarrow (r,\theta,z)$

- (r,θ) are the polar coordinates of (x,y).

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \qquad \mathrm{d}z\mathrm{d}y\mathrm{d}x = r\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

(Spherical coordinates) $(x,y,z) \longleftrightarrow (\rho,\phi,\theta)$

- $\rho \in [0,\infty)$ is the radial distance,
- $\phi \in [0,\pi]$ is the polar angle (angle measured from the z-axis), [pointing to the "pole"/zenith]
- $\theta \in [0,2\pi)$ is the azimuthal angle in the x-y-plane.

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi, \qquad \mathrm{d}z\mathrm{d}y\mathrm{d}x = \rho^2\sin\phi\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

Especially compared to physics, the order of polar and azimuthal angle often get swapped.

Example 120. What are the spherical coordinates for the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$?

Solution. The spherical coordinates for $(1,0,0)$ are $\rho=1$, $\phi=\pi/2$, $\theta=0$.

The spherical coordinates for $(0,1,0)$ are $\rho=1$, $\phi=\pi/2$, $\theta=\pi/2$.

The spherical coordinates for $(0,0,1)$ are $\rho=1$, $\phi=0$, $\theta=0$. (Here, θ could be anything. Why?! In that case, it is customary to choose $\theta=0$; as we did.)

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