Example 121. In cylindrical coordinates, what is described by $r=2$?

Solution. $r=2$ describes a cylinder (of radius 2, and axis the z-axis). Hence, **cylindrical coordinates**!

Example 122. In spherical coordinates, what is described by each of $\rho=2$, $\phi=\frac{\pi}{4}$, $\theta=\frac{\pi}{3}$?

What is described by $\phi=\frac{\pi}{2}$?

Solution. $\rho=2$ describes a sphere (of radius 2, centered at the origin). Hence, **spherical coordinates**!

$\phi=\pi/4$ describes a cone (opening up from the origin, making an angle of $\pi/4$ radians with the positive z-axis). In the extreme case $\phi=\pi/2$, the "cone" is the full xy-plane.

$\theta=\pi/3$ describes a half-plane (hinged along the z-axis, making a $\pi/3$ angle with the positive x-axis).

Example 123. Write down an integral for the volume of the ball $x^2+y^2+z^2\leqslant R^2$ using cartesian, spherical and cylindrical coordinates.

Solution.

(a) $\displaystyle\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ (done last class; not fun to compute this integral at all!)

(b) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta=\left(\int_0^R\rho^2\mathrm{d}\rho\right)\left(\int_0^{\pi}\sin\phi\,\mathrm{d}\phi\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right)=\frac{R^3}{3}\cdot(1+1)\cdot 2\pi=\frac{4\pi R^3}{3}$

Important. Make sure you understand exactly why, in this case, we were able to split the triple integral up!!

(c) In cylindrical coordinates, the equation for the ball becomes $r^2+z^2\leqslant R^2$.

$$\int_0^{2\pi}\int_0^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta=2\pi\int_0^R r\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\mathrm{d}z\,\mathrm{d}r=4\pi\int_0^R r\sqrt{R^2-r^2}\,\mathrm{d}r=4\pi\left[-\frac{(R^2-r^2)^{3/2}}{3}\right]_0^R=\frac{4\pi R^3}{3}$$

Example 124. What is the average value of $f(x,y,z)=z$ over the ball $x^2+y^2+z^2\leqslant 4$?

Solution. Recall that this average is $\frac{1}{\mathrm{vol}(D)}\iiint_D z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ where D is the ball of radius 2. Since we discussed this ball in the previous example, we already know that $\mathrm{vol}(D)=\frac{32\pi}{3}$ and we know how to write down a triple integral using spherical coordinates: $\mathrm{avg}=\frac{3}{32\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^2(\rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta=\ldots=0$.

Comment. This average "height" is the z-coordinate of the centroid (or center of mass, if our ball has uniform density). Since the ball D is symmetric about the xy-plane, the average value of z over D has to be 0!

Example 125. (HW!) What is the average value of $f(x,y,z)=z$ over the region D?

(a) If D is the cube $0\leqslant x,y,z\leqslant 2$.

(b) If D is the part of the ball $x^2+y^2+z^2\leqslant 4$ in the positive octand $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$.

(c) If D is the tetrahedron described by $x+y+z\leqslant 2$, $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$?

Solution. (Ignore until you have done the problem yourself.)

(a) $\mathrm{vol}(D)=8$, $\mathrm{avg}=\frac{1}{8}\int_0^2\int_0^2\int_0^2 z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x=\ldots=1$ [Why is that answer obvious in this simple case?]

(b) $\mathrm{vol}(D)=\frac{1}{8}\cdot\frac{4}{3}\pi 2^3=\frac{4\pi}{3}\approx 4.189$, $\mathrm{avg}=\frac{3}{4\pi}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2(\rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta=\ldots=\frac{3}{4}$

Note the (possibly unexpectedly) small volume compared to the cube!

(c) $\mathrm{vol}(D)=\frac{1}{3}\cdot\left(\frac{1}{2}2\cdot 2\right)\cdot 2=\frac{4}{3}$, $\mathrm{avg}=\frac{3}{4}\int_0^2\int_0^{2-x}\int_0^{2-x-y}z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x=\ldots=\frac{1}{2}$

Alternatively, you can find $\mathrm{vol}(D)$ as $\mathrm{vol}(D)=\int_0^2\int_0^{2-x}\int_0^{2-x-y}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x=\ldots=\frac{4}{3}$.

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