Line integrals

Example 126. (review) Find a parametrization for the line segment from $(1,2,3)$ to $(1,1,1)$.

Solution. For instance, $\boldsymbol{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$ with $t \in [0,1]$.

Example 127. (review) Find a parametrization for the upper half of the circle $x^2 + y^2 = 4$.

Solution. For instance, $\boldsymbol{r}(t) = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}$, $t \in [0, \pi]$.

Solution. Or, for instance, $\boldsymbol{r}(t) = \begin{bmatrix} t \\ \sqrt{4-t^2} \end{bmatrix}$, $t \in [-2,2]$.

Suppose that the curve C is parametrized by $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, with parameter $t \in [a,b]$. The **arc length** of the curve is

$$\text{length}(C) = \int_C \mathrm{d}s = \int_a^b \|\boldsymbol{r}'(t)\| \mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \mathrm{d}t.$$

We have seen all this, except the **line integral** $\int_C \mathrm{d}s$. The idea is that $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ is the length of a tiny piece of the curve C and we are adding these over the entire curve C to get the length of C.
For general line integrals $\int_C f(x,y,z)\mathrm{d}s$, we proceed in the same way, but instead of just adding up little lengths $\mathrm{d}s$ along the curve C we add up $f(x,y,z)\mathrm{d}s$ along the curve C.

Line integrals (with respect to arc length): $\displaystyle\int_C f(x,y,z)\mathrm{d}s = \int_a^b f(x(t), y(t), z(t)) \|\boldsymbol{r}'(t)\| \mathrm{d}t$

Like in the case of arc length, the choice of parametrization for C does not matter.

Example 128. Evaluate $\displaystyle\int_C \mathrm{d}s$, $\displaystyle\int_C x\mathrm{d}s$ and $\displaystyle\int_C xy\mathrm{d}s$ where C is the straight-line segment from $(0,1)$ to $(1,0)$.

Solution. We use $\boldsymbol{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t \\ 1-t \end{bmatrix}$, $t \in [0,1]$. Then, $\|\boldsymbol{r}'(t)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

$\displaystyle\int_C \mathrm{d}s = \int_0^1 \sqrt{2}\mathrm{d}t = \sqrt{2}$ (the arc length of C), $\displaystyle\int_C x\mathrm{d}s = \int_0^1 t\sqrt{2}\mathrm{d}t = \sqrt{2}\left[\frac{t^2}{2}\right]_0^1 = \frac{\sqrt{2}}{2}$

$\displaystyle\int_C xy\mathrm{d}s = \int_0^1 t(1-t)\sqrt{2}\mathrm{d}t = \sqrt{2}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{\sqrt{2}}{6}$

Example 129. Evaluate $\displaystyle\int_C x\mathrm{d}s$ where C is the graph of $y = 1 - x^2$ from $(0,1)$ to $(1,0)$.

Solution. We use the parametrization $\boldsymbol{r}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 1-t^2 \end{bmatrix}$, $t \in [0,1]$. Then, $\|\boldsymbol{r}'(t)\| = \sqrt{1 + (-2t)^2}$.

$$\int_C x\mathrm{d}s = \int_0^1 t\sqrt{1+4t^2}\mathrm{d}t = \frac{1}{8}\int_1^5 \sqrt{u}\mathrm{d}u = \left[\frac{1}{8}\frac{u^{3/2}}{3/2}\right]_1^5 = \frac{5^{3/2}-1}{12}$$

Comment. Without any computations it is clear (why?!) that $\displaystyle\int_C x\mathrm{d}s > \int_L x\mathrm{d}s$ where L is the straight-line segment from $(0,1)$ to $(1,0)$ (as in the previous example). Indeed, $\dfrac{5^{3/2}-1}{12} \approx 0.848 > \dfrac{\sqrt{2}}{2} \approx 0.707$.

Armin Straub
straub@southalabama.edu