Review.

- **(FToLI)** If C is a curve from A to B, then $\int_C \nabla f \cdot \mathrm{d}\boldsymbol{r} = f(B) - f(A)$.

  Actually, the proof is simple: by the chain rule, $\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{r}(t)) = \nabla f \cdot \boldsymbol{r}'(t)$. Hence, with the usual notations,
  $\int_C \nabla f \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \nabla f \cdot \boldsymbol{r}'(t)\mathrm{d}t = \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{r}(t))\right]\mathrm{d}t = \left[f(\boldsymbol{r}(t))\right]_a^b = f(B) - f(A)$.

- $\boldsymbol{F}$ is **conservative** if the line integral $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is **path independent**.

  Comment. $\boldsymbol{F}$ is conservative if and only if $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$ for any loop C. Can you explain why?

- $\boldsymbol{F}$ is a conservative field if and only if $\boldsymbol{F}$ is a gradient field.

Example 141. Visualize the vector fields $\boldsymbol{F} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{G} = \begin{bmatrix} -y \\ x \end{bmatrix}$.

Solution. A vector field $\boldsymbol{F}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$ or $\boldsymbol{F}(x, y, z) = \begin{bmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{bmatrix}$ assigns a vector to each point.

Therefore, one way to sketch vector fields is to sketch, for several points, this corresponding vector (drawn with its tail at the point). See Figures 15.11 and 15.12 in our book for such sketches of the "radial field" $\boldsymbol{F} = \begin{bmatrix} x \\ y \end{bmatrix}$ and the "spin field" $\boldsymbol{G} = \begin{bmatrix} -y \\ x \end{bmatrix}$.

(Component Test) How to check if $\boldsymbol{F}$ is conservative on a **simply connected** region D?

$\boldsymbol{F}(x, y) = \begin{bmatrix} M(x, y) \\ N(x, y) \end{bmatrix}$ is conservative if and only if $M_y = N_x$.

$\boldsymbol{F}(x, y, z) = \begin{bmatrix} M(x, y, z) \\ N(x, y, z) \\ P(x, y, z) \end{bmatrix}$ is conservative if and only if $M_y = N_x$, $M_z = P_x$, $N_z = P_y$.

Fine print: $\boldsymbol{F}$ needs to have continuous partial derivatives on the open set D.

D is simply connected if every loop in D can be contracted to a point (without leaving D). Figure 15.22

Example 142. Are the vector fields $\boldsymbol{F} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{G} = \begin{bmatrix} -y \\ x \end{bmatrix}$ conservative?

Solution. (direct approach)

(a) Let's try to find a potential function f (i.e. an f such that $\boldsymbol{F} = \nabla f$). Note that $f_x = x$ implies that $f = \frac{1}{2}x^2 + C(y)$. (Likewise, $f_y = y$ implies that $f = \frac{1}{2}y^2 + C(x)$. So you get also start with this.) Then $f_y = C'(y)$ which we need to match with $f_y = y$. Thus $C(y) = \frac{1}{2}y^2 + D$. Combined, we find that $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$ is a potential function for $\boldsymbol{F}$. In particular, $\boldsymbol{F}$ is conservative.

(b) Let's try to find g such that $\boldsymbol{G} = \nabla g$. Note that $g_x = -y$ implies that $g = -yx + C(y)$. Then $g_y = -x + C'(y)$ which we need to match with $g_y = x$. This is impossible! There is no potential function for $\boldsymbol{G}$. In particular, $\boldsymbol{G}$ is not conservative.

Solution. (component test) We check whether $M_y = N_x$.

(a) $M_y = \frac{\partial}{\partial y} x = 0$ and $N_x = \frac{\partial}{\partial x} y = 0$ are equal. Hence, $\boldsymbol{F}$ is conservative on $\mathbb{R}^2$ (which is simply connected).

(b) $M_y = \frac{\partial}{\partial y}(-y) = -1$ and $N_x = \frac{\partial}{\partial x} x = 1$ are not equal. Hence, $\boldsymbol{G}$ is not conservative.

Example 143. Are the vector fields $\boldsymbol{F} = \begin{bmatrix} y \\ x \end{bmatrix}$, $\boldsymbol{G} = \begin{bmatrix} 1 - 2xy \\ 1 + x^2 \end{bmatrix}$, $\boldsymbol{K} = \begin{bmatrix} 1 - 2xy \\ 1 - x^2 \end{bmatrix}$ and $\boldsymbol{H} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ conservative? If so, determine a potential function. [Yes: xy, No, Yes: $x - x^2y + y$, Yes: $\frac{1}{2}(x^2 + y^2 + z^2)$]