Example 146. (physical work)

- $W = Fd$ (work done moving an object a distance d; constant force of magnitude F in direction of motion)
- $W = \int_a^b F(x)\mathrm{d}x$ (work done moving an object from a to b; variable force F in direction of motion)
- $W = \boldsymbol{F}\cdot\boldsymbol{D}$ (work done moving an object along $\boldsymbol{D}$; constant force $\boldsymbol{F}$)

  Why? The effective force is the projection of $\boldsymbol{F}$ onto $\boldsymbol{D}$. Its magnitude is $\|\boldsymbol{F}\|\cos(\theta)$, where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{D}$ (make a sketch!). Hence, $W = \|\boldsymbol{F}\|\cos(\theta)\|D\| = \boldsymbol{F}\cdot\boldsymbol{D}$. (Also, see Section 11.3.)
- $W = \int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ (work done moving an object along the curve C; variable force $\boldsymbol{F}$)

  Why? Recall that $\mathrm{d}\boldsymbol{r} = \boldsymbol{r}'\mathrm{d}t$ is a small displacement along the curve C. Therefore, $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ is the work needed for that small displacement. It remains to add all these up.

Example 147. Find the work done by $\boldsymbol{F}$ moving an object from $(3,0)$ to $(0,3)$ along the circle C of radius 3 (counterclockwise) if **(a)** $\boldsymbol{F} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, **(b)** $\boldsymbol{F} = \begin{bmatrix} -y \\ x \end{bmatrix}$.

Solution. In both cases, the work is $W = \int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$.

(a) Computing the line integral directly: $W = \int_C \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\mathrm{d}\boldsymbol{r} = \int_0^{\pi/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix}\mathrm{d}t = \int_0^{\pi/2} 3\cos t\,\mathrm{d}t = 3.$

However, it is easier to notice that $\boldsymbol{F}$ is conservative (indeed, $M_y = N_x$) with potential $f(x,y) = y$.

Therefore, by the Fundamental Theorem, $W = \int_C \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\mathrm{d}\boldsymbol{r} = \Big[y\Big]_{(3,0)}^{(0,3)} = 3.$

Comment. (Ignoring signs) we can think of $\boldsymbol{F}$ as a gravitational field. As you probably know from physics, gravity is a **conservative force**; meaning, the work done in moving a particle between two points is independent of the path taken. Indeed, this work can be obtained as the change in **potential energy**, which is exactly what we compute when using the Fundamental Theorem.

(b) $W = \int_C \begin{bmatrix} -y \\ x \end{bmatrix}\cdot\mathrm{d}\boldsymbol{r} = \int_0^{\pi/2} \begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix}\cdot\begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix}\mathrm{d}t = \int_0^{\pi/2} 9\mathrm{d}t = \frac{9\pi}{2}$

Note that $\boldsymbol{F}$ is not conservative ($M_y \neq N_x$). Hence, we cannot use the Fundamental Theorem.

Comment. In this very simple example, we can actually see that answer directly: observe that the force $\boldsymbol{F}$ has constant magnitude 3 on C (why?) and it always is in the direction of motion (sketch the vector field $\boldsymbol{F}$!). Since, the total distance is $\frac{1}{4}(2\pi\cdot 3) = \frac{3\pi}{2}$, the total work is $W = Fd = 3\cdot\frac{3\pi}{2} = \frac{9\pi}{2}$.

Review. $\boldsymbol{F} = \begin{bmatrix} M \\ N \end{bmatrix}$ is conservative on a simply connected region D if and only if $M_y = N_x$.

Theorem 148. (Green's Theorem) Let R be a 2D region enclosed by a simple loop C, oriented counterclockwise. $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$.

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \iint_R (N_x - M_y)\mathrm{d}x\mathrm{d}y$$

Fine print: C piecewise smooth; M, N need to have continuous partial derivatives in an open region containing R.

What happens if $\boldsymbol{F}$ is conservative?

- Then, $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = 0$ by the Fundamental Theorem of Line Integrals.
- Also, $\iint_R (N_x - M_y)\mathrm{d}x\mathrm{d}y = 0$ because $N_x - M_y = 0$.

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