- $\oint_C$ means the same as $\int_C$ but emphasizes that C is a simple loop (positively oriented).

  Simple means that the curve C does not cross itself.

Example 149. Evaluate $\oint_C \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathrm{d}\boldsymbol{r} = \oint_C -y\mathrm{d}x + x\mathrm{d}y$ where C is the unit circle, oriented positively (that is, counterclockwise).

Solution. (directly) $W = \oint_C \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathrm{d}\boldsymbol{r} = \int_0^{2\pi} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \mathrm{d}t = \int_0^{2\pi} 1\mathrm{d}t = 2\pi$

Solution. (Green) $W = \oint_C \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \mathrm{d}\boldsymbol{r} = \iint_R (N_x - M_y)\mathrm{d}x\mathrm{d}y = \iint_R 2\mathrm{d}x\mathrm{d}y = 2\mathrm{area}(R) = 2\pi$

Example 150. (planimeter) If we choose $\boldsymbol{F}$ such that $N_x - M_y = 1$, then Green's Theorem becomes $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_R (N_x - M_y)\mathrm{d}x\mathrm{d}y = \mathrm{area}(R)$.

One such choice is $\boldsymbol{F} = \frac{1}{2}\begin{bmatrix} -y \\ x \end{bmatrix}$ as in the previous example. Another simple choice is $\boldsymbol{F} = \begin{bmatrix} 0 \\ x \end{bmatrix}$.

These choices lead to the interesting formulas $\mathrm{area}(R) = \oint_C x\mathrm{d}y$ and $\mathrm{area}(R) = \frac{1}{2}\oint_C (-y\mathrm{d}x + x\mathrm{d}y)$.

A **planimeter** is a measuring instrument used to determine the area of an arbitrary two-dimensional shape; it works based on the principle we just observed!

Example 151. Evaluate $\oint_C xy\mathrm{d}x + x^2y\mathrm{d}y$ where C is the positively oriented boundary of the triangle T with vertices $(0,0)$, $(1,0)$, $(1,2)$.

Solution. (directly, HW!) We parametrize the three sides as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$, from $t=0$ to $t=1$, as well as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}$, from $t=0$ to $t=2$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \end{bmatrix}$, from $t=0$ to $t=1$ (we need to be careful about the orientation here!). Then, $\oint_C xy\mathrm{d}x + x^2y\mathrm{d}y$

$= \int_0^1 0 \cdot 1\mathrm{d}t + 0 \cdot 0\mathrm{d}t + \int_0^2 t \cdot 0\mathrm{d}t + t \cdot 1\mathrm{d}t + \int_0^1 (1-t)(2-t)\cdot(-1)\mathrm{d}t + (1-t)^2(2-t)\cdot(-2)\mathrm{d}t = 0 + 2 - \frac{5}{3} = \frac{1}{3}.$

Solution. (Green) Note that the triangle can be described by the bounds $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2x$.

$\oint_C xy\mathrm{d}x + x^2y\mathrm{d}y = \iint_T (2xy - x)\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^{2x} (2xy - x)\mathrm{d}y\mathrm{d}x$

$= \int_0^1 \left[xy^2 - xy\right]_{y=0}^{y=2x} \mathrm{d}x = \int_0^1 (4x^3 - 2x^2)\mathrm{d}x = \left[x^4 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}$

- Line integrals are also written as $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\mathrm{d}s$, where $\boldsymbol{T}$ is a unit vector tangent to the curve C (at each point).

  Why? Note that $\mathrm{d}\boldsymbol{r} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$ is a tangent vector (recall that $\mathrm{d}\boldsymbol{r} = \boldsymbol{r}'(t)\mathrm{d}t$ and that the velocity vector $\boldsymbol{r}'(t)$ is tangent to the curve C) of length $\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}s$. Hence, $\mathrm{d}\boldsymbol{r} = \boldsymbol{T}\mathrm{d}s$.

Armin Straub
straub@southalabama.edu