Think, again, of $\boldsymbol{F}$ as indicating the velocity of a fluid flowing.

- $\int_C \boldsymbol{F}\cdot\boldsymbol{T}\mathrm{d}s = \int_C M\mathrm{d}x + N\mathrm{d}y$ is the "flow of $\boldsymbol{F}$ along the curve C".

  If C is a loop then this is the circulation of $\boldsymbol{F}$ around C.

  Recall that $\boldsymbol{T}$ is a unit tangent vector, and $\boldsymbol{T}\mathrm{d}s = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$.

- $\int_C \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}s = \int_C M\mathrm{d}y - N\mathrm{d}x$ is the "flux of $\boldsymbol{F}$ across C".

  That is, the rate at which the fluid is entering or leaving the region enclosed by C.

  Note that, since $\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$ is tangent to C, the vector $\begin{bmatrix} \mathrm{d}y \\ -\mathrm{d}x \end{bmatrix}$ is normal (and pointing outwards if motion is counterclockwise) of length $\sqrt{\mathrm{d}y^2 + (-\mathrm{d}x)^2} = \mathrm{d}s$. Hence, $\boldsymbol{n}\mathrm{d}s = \begin{bmatrix} \mathrm{d}y \\ -\mathrm{d}x \end{bmatrix}$.

Theorem 155. (Green's Theorem, curl and divergence form) Let R be a 2D region enclosed by a simple loop C, oriented counterclockwise. $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$.

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}s = \iint_R \underbrace{\operatorname{div}\boldsymbol{F}}_{=M_x+N_y}\,\mathrm{d}x\mathrm{d}y, \qquad \oint_C \boldsymbol{F}\cdot\boldsymbol{T}\mathrm{d}s = \iint_R \underbrace{(\operatorname{curl}\boldsymbol{F})\cdot\boldsymbol{k}}_{=N_x-M_y}\,\mathrm{d}x\mathrm{d}y$$

Fine print: C piecewise smooth; M, N have continuous partial derivatives in an open region containing R.

The divergence form states that the outward flux of $\boldsymbol{F}$ across C can be computed by adding up all sources and sinks inside the region.

The curl form says that microscopic rotation adds up to the overall circulation of $\boldsymbol{F}$ around C.

Both interpretations of Green's Theorem generalize to 3D (in very different ways):

Theorem 156. (Stokes' Theorem) Let S be a surface enclosed by a simple loop C.

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{T}\mathrm{d}s = \iint_S (\operatorname{curl}\boldsymbol{F})\cdot\boldsymbol{n}\,\mathrm{d}\sigma$$

Fine print: C piecewise smooth; components of $\boldsymbol{F}$ have continuous partial derivatives in an open region containing S.

- Note that we get Green's Theorem as a special case of Stokes' Theorem.
- Just like line integrals $\mathrm{d}s = \|\boldsymbol{r}'\|\mathrm{d}t$, the surface integrals $\mathrm{d}\sigma$ can be computed using a parametrization $\boldsymbol{r}(t,u)$ of the surface. Indeed, $\mathrm{d}\sigma = \|\boldsymbol{r}_t \times \boldsymbol{r}_u\|\mathrm{d}t\mathrm{d}u$.

Recall that $\|\boldsymbol{r}_t \times \boldsymbol{r}_u\|$ is the area of the parallelogram with sides $\boldsymbol{r}_t$ and $\boldsymbol{r}_u$.

Theorem 157. (Divergence Theorem) Let S be a (closed) surface enclosing a region D.

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma = \iiint_D \operatorname{div}\boldsymbol{F}\,\mathrm{d}V$$

Fine print: S piecewise smooth; components of $\boldsymbol{F}$ have continuous partial derivatives in an open region containing D.

Concluding comments. For all of these integral theorems (Green's Theorem, Stokes' Theorem, Divergence Theorem) the integral of a differential operator acting on a field over a region R equals an appropriate sum of the field components over the boundary of R. In fact, in the language of differential forms, all of them are subsumed by the (generalized) Stokes' Theorem

$$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$$

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