Preparing for Midterm #1

MATH 126 — Calculus III
Friday, Feb 19

Please print your name:

Problem 1. Consider the points $P=(1,2,-1)$, $Q=(2,3,3)$ and $R=(-1,1,2)$.

(a) Determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

(b) Find the distance between P and Q.

(c) Find a parametrization for the line through P and Q.

(d) Find a parametrization for the line segment from Q to R.

(e) Find an equation for the plane through P, Q and R.

(f) Find the distance between P and the line through Q and R.

(g) Find an equation for the sphere with center P and radius 5.

(h) Find the area of the triangle with vertices P, Q and R.

Solution.

(a) $\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\overrightarrow{QR} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$

(b) $\left\| \overrightarrow{PQ} \right\| = \left\| \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\| = \sqrt{1+1+16} = \sqrt{18}$

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+t \\ 2+t \\ -1+4t \end{bmatrix}$ with $t \in (-\infty, \infty)$

(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-3t \\ 3-2t \\ 3-t \end{bmatrix}$ with $t \in [0,1]$

(e) $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\overrightarrow{QR} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$ are parallel to the plane.

Hence, we find a normal vector for the plane as $\boldsymbol{n} = \overrightarrow{PQ} \times \overrightarrow{QR} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \times \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1-(-8) \\ -12-(-1) \\ -2-(-3) \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \\ 1 \end{bmatrix}$.

At this point, we know that the plane can be described by an equation of the form $7x - 11y + z = d$, and it remains to find the value d.

Since the point $P=(1,2,-1)$ is on the plane, we have $7 \cdot 1 - 11 \cdot 2 + (-1) = d$, that is, $d = -16$.

In conclusion, our plane is described by the equation $7x - 11y + z = -16$.

Suggestion: If you have time left, verify this equation by plugging in all three points P, Q, R. If they all satisfy this equation, then your equation is guaranteed to be correct!

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Comment: Here is a slightly longer derivation, which explains why we get an equation of the above shape. Our plane is known to contain the point $P=(1,2,-1)$. On the other hand, another point $S=(x,y,z)$ is on the plane if and only if $\overrightarrow{PS}\cdot\boldsymbol{n}=0$. (Why?)

Spelling out this last equation, we get $\begin{bmatrix} x-1 \\ y-2 \\ z+1 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -11 \\ 1 \end{bmatrix}=0$, or $7(x-1)-11(y-2)+(z+1)=0$, which simplifies to $7x-11y+z=-16$. (And note how we actually ended up doing the same computation!)

(f) Recall that the distance between a point A and the line through B with direction $\boldsymbol{v}$ is $d=\frac{\|\overrightarrow{BA}\times\boldsymbol{v}\|}{\|\boldsymbol{v}\|}$.

For the distance between P and the line through Q and R, we therefore get

$$d=\frac{\left\|\overrightarrow{QP}\times\overrightarrow{QR}\right\|}{\left\|\overrightarrow{QR}\right\|}=\frac{\left\|\begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}\times\begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}\right\|}{\left\|\begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}\right\|}=\frac{\left\|\begin{bmatrix} 1-8 \\ 12-1 \\ 2-3 \end{bmatrix}\right\|}{\sqrt{9+4+1}}=\frac{\left\|\begin{bmatrix} -7 \\ 11 \\ -1 \end{bmatrix}\right\|}{\sqrt{14}}=\frac{\sqrt{49+121+1}}{\sqrt{14}}=\sqrt{\frac{171}{14}}.$$

Suggestion: If you have time to review your answers, check each cross product by verifying that the result is orthogonal to both of the original vectors. For instance, here $\begin{bmatrix} -7 \\ 11 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}=0$ and $\begin{bmatrix} -7 \\ 11 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}=0$.

(g) $(x-1)^2+(y-2)^2+(z+1)^2=5^2$

Extra detail: A point $A=(x,y,z)$ is on that sphere if and only if $\left\|\overrightarrow{PA}\right\|=5$. The equation above is the expanded version of $\left\|\overrightarrow{PA}\right\|^2=5^2$.

(h) Let $\boldsymbol{v}=\overrightarrow{PQ}=\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{w}=\overrightarrow{PR}=\begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$ be two sides of our triangle (you can pick other sides, no problem!).

Then, borrowing from what we learned in class, the area of the triangle is

$$\frac{1}{2}\|\boldsymbol{v}\times\boldsymbol{w}\|=\frac{1}{2}\left\|\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\times\begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}\right\|=\frac{1}{2}\left\|\begin{bmatrix} 3-(-4) \\ -8-3 \\ -1-(-2) \end{bmatrix}\right\|=\frac{1}{2}\left\|\begin{bmatrix} 7 \\ -11 \\ 1 \end{bmatrix}\right\|=\frac{1}{2}\sqrt{49+121+1}=\frac{\sqrt{171}}{2}.$$

□

Problem 2. Consider the vectors $\boldsymbol{v}_1=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{v}_2=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_3=\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

(a) Determine $|\boldsymbol{v}_1|$.

(b) Determine $\boldsymbol{v}_1\cdot\boldsymbol{v}_2$.

(c) Determine $\boldsymbol{v}_1\times\boldsymbol{v}_2$.

(d) What is the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$? (Between $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$?)

(e) Determine the projection of $\boldsymbol{v}_1$ onto $\boldsymbol{v}_2$. (The projection of $\boldsymbol{v}_2$ onto $\boldsymbol{v}_3$. Explain your answer!)

(f) Find a vector which has length 3 and is parallel to $\boldsymbol{v}_1$.

(g) Are there two of the three vectors which are parallel? Perpendicular?

(h) Find a parametrization for the line through $(1, -1, 2)$ which is parallel to $\boldsymbol{v}_1$.

(i) Find an equation for the plane through $(1, 2, 3)$ which is perpendicular to $\boldsymbol{v}_1$.

(j) Find an equation for the plane through $(1, 2, 3)$ which is parallel to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Solution.

(a) $\|\boldsymbol{v}_1\| = \left\| \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\| = \sqrt{1+4+1} = \sqrt{6}$

(b) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 + 2 - 1 = 1$

(c) $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-(-1) \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

(d) For the angle α between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we have $\cos\alpha = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_1\| \|\boldsymbol{v}_2\|} = \frac{1}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{12}}$ and hence $\alpha = \arccos\left(\frac{1}{\sqrt{12}}\right) \approx 1.278$ (which is about $73.22°$).

For the angle β between $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, we have $\cos\beta = \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_3}{\|\boldsymbol{v}_2\| \|\boldsymbol{v}_3\|} = 0$ and hence $\beta = \frac{\pi}{2}$ (that is, $90°$). In other words, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are perpendicular.

(e) The projection of $\boldsymbol{v}_1$ onto $\boldsymbol{v}_2$ is $\text{proj}_{\boldsymbol{v}_2} \boldsymbol{v}_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2} \boldsymbol{v}_2 = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

The projection of $\boldsymbol{v}_2$ onto $\boldsymbol{v}_3$ is $\text{proj}_{\boldsymbol{v}_3} \boldsymbol{v}_2 = \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_3}{\|\boldsymbol{v}_3\|^2} \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This is because $\boldsymbol{v}_2$ is perpendicular to $\boldsymbol{v}_3$.

(f) The vector $3 \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{3}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ has length 3 and is parallel to $\boldsymbol{v}_1$.

(g) Two vectors $\boldsymbol{a}, \boldsymbol{b}$ are parallel if and only if one of the vectors is a multiple of the other. The three vectors are clearly not multiples of each other.

Comment: You might also recall from class that two vectors $\boldsymbol{a}, \boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$. So you could also compute these cross products. That takes more time though.

Two vectors $\boldsymbol{a}, \boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. We have $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1$, $\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 6$ and $\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0$. Hence, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are perpendicular.

(h) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+t \\ -1+2t \\ 2-t \end{bmatrix}$ with $t \in (-\infty, \infty)$

(i) We know that the plane has normal vector $\boldsymbol{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and so can be described by an equation of the form $x + 2y - z = d$. To find the value d, we use the fact that the point $P = (1, 2, 3)$ is on the plane: $1 + 2 \cdot 2 - 3 = d$, and so, $d = 2$. Our plane is described by $x + 2y - z = 2$.

(j) The plane is parallel to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ if and only if it is perpendicular to $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

Hence, the plane can be described by an equation of the form $3x - y + z = d$. To find the value d, we use the fact that the point $P = (1, 2, 3)$ is on the plane: $3 \cdot 1 - 2 + 3 = d$, and so, $d = 4$. Our plane is described by $3x - y + z = 4$. □

Problem 3. Consider the plane described by $3x - y - 2z = 4$.

(a) Determine a normal vector for the plane.

(b) Find the x-intercept, the y-intercept and the z-intercept of the plane.

(c) Find a unit vector perpendicular to our plane.

(d) Find an equation for the plane through $(1,2,3)$ which is parallel to our plane.

(e) Is the line with parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ 2+t \\ 3+t \end{bmatrix}$ parallel to our plane?

(f) If possible, find the intersection of the line with parametrization $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-2t \\ 3+t \\ 1 \end{bmatrix}$ and our plane.

(g) Determine the distance between the point $(1,2,3)$ and our plane.

(h) Find (a parametrization for) the line in which our plane intersects with the plane $x + y + z = 2$.

Solution.

(a) $\boldsymbol{n} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ is a normal vector (as is any multiple of it).

(b) The intersection with x-axis, y-axis and z-axis are $\left(\frac{4}{3}, 0, 0\right)$, $(0, -4, 0)$ and $(0, 0, -2)$, respectively. The x-intercept is $\frac{4}{3}$, the y-intercept is -4 and the z-intercept is -2.

(c) There is two unit vectors perpendicular to our plane: $\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $-\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{-1}{\sqrt{14}} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

(d) Observe that planes are parallel if and only if they share the same normal direction. Hence, any plane parallel to $3x - y - 2z = 4$ can be described by an equation of the form $3x - y - 2z = d$. To find the value d, we use the fact that the point $P = (1,2,3)$ is on the plane: $3 \cdot 1 - 2 - 2 \cdot 3 = d$, and so, $d = -5$. Thus, our plane is described by $3x - y - 2z = -5$.

(e) A line with direction $\boldsymbol{v}$ is parallel to a plane with normal vector $\boldsymbol{n}$ if and only if $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = 0$, the given line is indeed parallel to our plane.

(f) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-2t \\ 3+t \\ 1 \end{bmatrix}$ is a point on the plane if and only if $3(1-2t) - (3+t) - 2 \cdot 1 = 4$. Simplified, this is $-2 - 7t = 4$, which we solve for $t = -\frac{6}{7}$. Hence, the line and plane intersect in the point $\begin{bmatrix} 1-2t \\ 3+t \\ 1 \end{bmatrix}_{t=-\frac{6}{7}} = \frac{1}{7}\begin{bmatrix} 19 \\ 15 \\ 7 \end{bmatrix}$.

Suggestion: Check that $\frac{1}{7}\begin{bmatrix} 19 \\ 15 \\ 7 \end{bmatrix}$ is indeed on the plane by plugging in.

(g) Recall that the distance between a point A and the plane through B with normal $\boldsymbol{n}$ is $d = \frac{|\overrightarrow{BA} \cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|}$.

For the distance between $A = (1,2,3)$ and our plane, for which we can choose $B = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ and $\boldsymbol{n} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, we get

$$d = \frac{|\overrightarrow{BA} \cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|} = \frac{\left|\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}\right|}{\left\|\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}\right\|} = \frac{|3 - 2 - 10|}{\sqrt{9+1+4}} = \frac{9}{\sqrt{14}}.$$

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(h) To find the intersection of $3x - y - 2z = 4$ and $x + y + z = 2$, we can subtract 3eq_2 from eq_1 (to eliminate x). $\text{eq}_1 - 3\text{eq}_2$ is $-4y - 5z = -2$. Since we cannot eliminate further, we now choose to set $z = t$, our parameter.

From $-4y - 5t = -2$, we then find $y = \frac{1}{2} - \frac{5}{4}t$.

Substituting into the first equation, we get $3x - \left(\frac{1}{2} - \frac{5}{4}t\right) - 2t = 4$, which we solve for $x = \frac{3}{2} + \frac{1}{4}t$.

Taken together, we have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{1}{4}t \\ \frac{1}{2} - \frac{5}{4}t \\ t \end{bmatrix}$, which is a parametrization for the line of intersection.

Suggestion: To verify our result, we can check that the direction $\boldsymbol{v} = \begin{bmatrix} \frac{1}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix}$ is perpendicular to both of the normal vectors. Indeed, $\begin{bmatrix} \frac{1}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = 0$ and $\begin{bmatrix} \frac{1}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$.

Alternative solution: In the spirit of the above "suggestion", we can find the direction of the line of intersection by computing $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - (-2) \\ -2 - 3 \\ 3 - (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$. This just 4 times the direction vector we found above.

To write down a parametrization, we still need to find a point on the line. (You can do this by making an arbitrary choice for one of x, y or z, and then solving the two equations $3x - y - 2z = 4$ and $x + y + z = 2$ for the two remaining unknowns.)

□

Problem 4. Consider the line L_1 parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - t \\ 1 + 3t \\ t \end{bmatrix}$.

(a) Find a parametrization for the line parallel to L_1 through the point $(1, 2, 3)$.

(b) What is the distance between the point $(1, 2, 3)$ and the line L_1?

(c) Let L_2 be a second line, parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + 2t \\ -6t \\ 1 - 2t \end{bmatrix}$. Are L_1 and L_2 parallel? Are they the same line?

(d) Let L_3 be a third line, parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + t \\ 2 - 2t \\ 5 - t \end{bmatrix}$. Do L_1 and L_3 intersect? If yes, find their intersection.

(e) Let L_4 be a fourth line, parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + t \\ 8 - 2t \\ 3 - t \end{bmatrix}$. Do L_1 and L_4 intersect? If yes, find their intersection.

Solution.

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - t \\ 2 + 3t \\ 3 + t \end{bmatrix}$ with $t \in (-\infty, \infty)$

(b) Recall that the distance between a point A and the line through B with direction $\boldsymbol{v}$ is $d = \frac{\|\overrightarrow{BA} \times \boldsymbol{v}\|}{\|\boldsymbol{v}\|}$.

For the distance between $A = (1, 2, 3)$ and our line through $B = (2, 1, 0)$ and direction $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$, we therefore get

$$d = \frac{\|\overrightarrow{BA} \times \boldsymbol{v}\|}{\|\boldsymbol{v}\|} = \frac{\left\| \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\|} = \frac{\left\| \begin{bmatrix} 1 - 9 \\ -3 - (-1) \\ -3 - (-1) \end{bmatrix} \right\|}{\sqrt{1 + 9 + 1}} = \frac{\left\| \begin{bmatrix} -8 \\ -2 \\ -2 \end{bmatrix} \right\|}{\sqrt{11}} = \frac{\sqrt{64 + 4 + 4}}{\sqrt{11}} = \sqrt{\frac{72}{11}}.$$

(c) We need to check whether the direction vectors $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -6 \\ -2 \end{bmatrix}$ and $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ are parallel. They clearly are because $\boldsymbol{v}_2 = -2\boldsymbol{v}_1$. Hence, the lines are parallel.

Are the lines the same? Since we already know that the lines are parallel, this is the case if and only if they have a point in common. L_1 obviously contains the point $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and so we check whether that point is also on L_2: however, there is no t such that $\begin{bmatrix} 1+2t \\ -6t \\ 1-2t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. (Why?) This means that the lines are not the same.

(d) To determine whether the two lines intersect, we ask whether there are values t_1 and t_2 such that

$$\begin{bmatrix} 2-t_1 \\ 1+3t_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} 1+t_2 \\ 2-2t_2 \\ 5-t_2 \end{bmatrix}.$$

We can use the first two equations $2-t_1 = 1+t_2$ and $1+3t_1 = 2-2t_2$ to solve for t_1 and t_2 (and then need to check whether these values also work in the third equation). We can eliminate t_1 by combining the equations as $3\mathrm{eq}_1 + \mathrm{eq}_2$ to get $7 = 5 + t_2$, which gives us $t_2 = 2$. Plugged into eq_1, we then find $2 - t_1 = 1 + 2$ and hence $t_1 = -1$. The third equation $t_1 = 5 - t_2$, however, does not hold for $t_1 = -1$ and $t_2 = 2$. This means that the two lines do not intersect. They are skew.

(e) To determine whether the two lines intersect, we ask whether there are values t_1 and t_2 such that

$$\begin{bmatrix} 2-t_1 \\ 1+3t_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} -1+t_2 \\ 8-2t_2 \\ 3-t_2 \end{bmatrix}.$$

We can use the first two equations $2-t_1 = -1+t_2$ and $1+3t_1 = 8-2t_2$ to solve for t_1 and t_2 (and then need to check whether these values also work in the third equation). We can eliminate t_1 by combining the equations as $3\mathrm{eq}_1 + \mathrm{eq}_2$ to get $7 = 5 + t_2$, which gives us $t_2 = 2$. Plugged into eq_1, we then find $2 - t_1 = -1 + 2$ and hence $t_1 = 1$. The third equation $t_1 = 3 - t_2$ does indeed hold for $t_1 = 1$ and $t_2 = 2$. This means that the two lines do intersect. The point of intersection is $\begin{bmatrix} 2-t_1 \\ 1+3t_1 \\ t_1 \end{bmatrix}_{t_1=1} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.

Suggestion: For verification, we can test that $\begin{bmatrix} -1+t_2 \\ 8-2t_2 \\ 3-t_2 \end{bmatrix}_{t_2=2} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ is indeed the same point.

□

Problem 5. Set up an integral (but do not evaluate) for the length of the curve $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-3t \\ \cos(2t) \\ t^2 \end{bmatrix}$ with $t \in [\pi, 2\pi]$.

Solution. The length of the curve is

$$\int_\pi^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t = \int_\pi^{2\pi} \sqrt{(-3)^2 + (-2\sin(2t))^2 + (2t)^2}\, \mathrm{d}t = \int_\pi^{2\pi} \sqrt{9 + 4\sin^2(2t) + 4t^2}\, \mathrm{d}t.$$

□

Problem 6. Find (a parametrization of) the tangent line to the curve $P(t) = \begin{bmatrix} (t+1)\ln t \\ (2t+1)^2 \\ t^3 \end{bmatrix}$ at $t = 1$.

Solution. The tangent line goes through $P(1) = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}$ and has direction $P'(1) = \begin{bmatrix} \ln t + (t+1)\frac{1}{t} \\ 4(2t+1) \\ 3t^2 \end{bmatrix}_{t=1} = \begin{bmatrix} 2 \\ 12 \\ 3 \end{bmatrix}$.

Hence, it is parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 12 \\ 3 \end{bmatrix} = \begin{bmatrix} 2t \\ 9+12t \\ 1+3t \end{bmatrix}$.

□

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