Midterm #1

MATH 227 — Calculus III
Friday, Feb 19

Please print your name:

Problem 1. What is the distance between the points $A=(2,0,1)$ and $B=(1,-1,1)$?

Solution. The distance is $\left\|\overrightarrow{AB}\right\| = \left\|\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right\| = \sqrt{2}$. □

Problem 2. Consider the vectors $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Determine the following:

(a) $|\boldsymbol{v}| =$

(b) $\boldsymbol{v} \cdot \boldsymbol{w} =$

(c) $\boldsymbol{v} \times \boldsymbol{w} =$

(d) $\mathrm{proj}_{\boldsymbol{v}}\, \boldsymbol{w} =$ (the projection of $\boldsymbol{w}$ onto $\boldsymbol{v}$)

Problem 3. Set up an integral (but do not evaluate) for the length of the curve $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-t \\ 1+t^2 \\ e^{3t} \end{bmatrix}$ with $t \in [0,1]$.

Solution. The length of the curve is

$$\int_0^1 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t = \int_0^1 \sqrt{(-1)^2 + (2t)^2 + (3e^{3t})^2}\, \mathrm{d}t = \int_0^1 \sqrt{1 + 4t^2 + 9e^{6t}}\, \mathrm{d}t.$$ □

Armin Straub
straub@southalabama.edu

Problem 4. Consider the triangle with vertices $P=(1,1,0)$, $Q=(3,1,1)$ and $R=(2,4,1)$.

(a) Find the area of the triangle with vertices P, Q and R.

(b) Find an equation for the plane through the points P, Q and R.

Solution.

(a) Let $\boldsymbol{v}=\overrightarrow{PQ}=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}=\overrightarrow{PR}=\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ be two sides of our triangle (you can pick other sides, no problem!).

Then, borrowing from what we learned in class, the area of the triangle is

$$\frac{1}{2}\|\boldsymbol{v}\times\boldsymbol{w}\|=\frac{1}{2}\left\|\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\times\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\|=\frac{1}{2}\left\|\begin{bmatrix} 0-3 \\ 1-2 \\ 6-0 \end{bmatrix}\right\|=\frac{1}{2}\left\|\begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix}\right\|=\frac{1}{2}\sqrt{9+1+36}=\frac{\sqrt{46}}{2}.$$

(b) The vectors $\boldsymbol{v}=\overrightarrow{PQ}=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}=\overrightarrow{PR}=\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ are parallel to the desired plane.

Hence, we find a normal vector for the plane as $\boldsymbol{n}=\boldsymbol{v}\times\boldsymbol{w}=\begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix}$.

At this point, we know that the plane can be described by an equation of the form $3x+y-6z=d$, and it remains to find the value d.

Since the point $P=(1,1,0)$ is on the plane, we have $3\cdot 1+1-6\cdot 0=d$, that is, $d=4$.

In conclusion, our plane is described by the equation $3x+y-6z=4$. □

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Problem 5. Find (a parametrization of) the tangent line to the curve $P(t) = \begin{bmatrix} e^{t^2} \\ \sin(2t) \\ t+2 \end{bmatrix}$ at $t = 0$.

Solution. The tangent line goes through $P(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and has direction $P'(0) = \begin{bmatrix} 2te^{t^2} \\ 2\cos(2t) \\ 1 \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Hence, it is parametrized by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2t \\ 2+t \end{bmatrix}$. □

Problem 6. Consider the plane described by $x - 2y + 2z = 4$.

(a) A normal vector for our plane is $\boldsymbol{n} = \boxed{}$.

(b) Find an equation for the plane through the point $(1, 2, 3)$ which is parallel to our plane.

(c) Determine the distance between the point $(1, 2, 3)$ and our plane.

Solution.

(a) A normal vector for our plane is $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

(b) Such a plane can be described by an equation of the form $x - 2y + 2z = d$. To find the value d, we use the fact that the point $P = (1, 2, 3)$ is on the plane: $1 - 2 \cdot 2 + 2 \cdot 3 = d$, and so, $d = 3$. Our plane is described by $x - 2y + 2z = 3$.

(c) Recall that the distance between a point A and the plane through B with normal $\boldsymbol{n}$ is $d = \frac{|\overrightarrow{BA} \cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|}$.

For the distance between $A = (1, 2, 3)$ and our plane, for which we can choose $B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ and $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, we get

$$d = \frac{|\overrightarrow{BA} \cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|} = \frac{\left|\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right|}{\left\|\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right\|} = \frac{|1 - 4 + 2|}{\sqrt{1 + 4 + 4}} = \frac{1}{3}.$$

□

Problem 7. **(Bonus!)** What kind of object is that? Do you know its name?

Solution. This is a fractal. This particular one is known as Koch's snowflake. □

Armin Straub
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