# Preparing for Midterm #2

**MATH 126 — Calculus III**
**Friday, Apr 1**

***Please print your name:***

**Problem 1.** Consider the function $f(x,y) = \dfrac{1}{1 + 2x^2 - xy}$.

(a) What is the natural domain of $f(x,y)$?

(b) Compute the partial derivatives $f_x$ and $f_{xy}$.

(c) Find the linearization of $f(x,y)$ at $(2,3)$.

(d) Compute the gradient $\nabla f$.

(e) Show that $f(x,y)$ is a solution to the partial differential equation $x\dfrac{\partial f}{\partial x} + (4x - y)\dfrac{\partial f}{\partial y} = 0$.

(f) Determine and sketch the level curve $f(x,y) = 1$.

(g) Find a vector which is orthogonal to the curve $f(x,y) = 1$ at the point $(1,2)$.

[Make sure to compare your answer to what you got for the level curve $f(x,y) = 1$.]

(h) Find the derivative of $f(x,y)$ at $(1,2)$ in direction $\boldsymbol{v} = 3\boldsymbol{i} + \boldsymbol{j}$.

(i) Find a vector which is orthogonal to the curve $f(x,y) = 2$ at the point $(1/2, 2)$.

(j) In which direction does $f(x,y)$ at $(1/2, 2)$ increase most rapidly?

(k) Find the equation for the plane tangent to the graph of $f(x,y)$ at $(1,2)$.

(l) Let $w = f(x,y)$ and $x = 2 + t$, $y = \cos(t)$. Find $\frac{\mathrm{d}w}{\mathrm{d}t}$ (in terms of $t$) in two ways:

- by expressing $w$ in terms of $t$ and differentiating directly,
- by using the chain rule.

(m) Find the local extreme values and saddles of $f(x,y)$.

**Solution.**

(a) $f(x,y)$ is well-defined for all $x$, $y$ such that $1 + 2x^2 - xy \neq 0$. In other words, the natural domain is $\{(x,y) \in \mathbb{R}^2 : 1 + 2x^2 - xy \neq 0\}$.

*Optional thoughts.* Note that the condition is equivalent to $y \neq \frac{1 + 2x^2}{x}$, or $y \neq \frac{1}{x} + 2x$. Hence, $f(x,y)$ is well-defined for all points $x, y$ in the plane except those on the curve $y = \frac{1}{x} + 2x$.

(b) $f_x = \dfrac{\partial}{\partial x} \dfrac{1}{1 + 2x^2 - xy} = \dfrac{y - 4x}{(1 + 2x^2 - xy)^2}$

$f_y = \dfrac{\partial}{\partial y} \dfrac{1}{1 + 2x^2 - xy} = \dfrac{x}{(1 + 2x^2 - xy)^2}$

Armin Straub
straub@southalabama.edu

$$f_{xy} = \frac{\partial}{\partial y}\frac{y-4x}{(1+2x^2-xy)^2} = \frac{1\cdot(1+2x^2-xy)^2-(y-4x)2(1+2x^2-xy)\cdot(-x)}{(1+2x^2-xy)^4} = \frac{1-6x^2+xy}{(1+2x^2-xy)^3}$$

(c) We have $f(2,3) = \frac{1}{3}$, $f_x(2,3) = -\frac{5}{9}$ and $f_y(2,3) = \frac{2}{9}$.

Hence, the linearization of $f(x,y)$ at $(2,3)$ is $L(x,y) = \frac{1}{3} - \frac{5}{9}(x-2) + \frac{2}{9}(y-3)$.

(d) $\nabla f = \begin{bmatrix} \frac{y-4x}{(1+2x^2-xy)^2} \\ \frac{x}{(1+2x^2-xy)^2} \end{bmatrix}$ $\quad$ ($\nabla f = \frac{1}{(1+2x^2-xy)^2}\begin{bmatrix} y-4x \\ x \end{bmatrix}$ is the same but looks a bit cleaner.)

(e) We have $x\frac{\partial f}{\partial x} + (4x-y)\frac{\partial f}{\partial y} = \frac{x(y-4x)}{(1+2x^2-xy)^2} + \frac{(4x-y)x}{(1+2x^2-xy)^2} = 0$.

(f) $\frac{1}{1+2x^2-xy} = 1$ is equivalent to $1+2x^2-xy = 1$, which further simplifies to $xy = 2x^2$. There is two solutions: $x=0$ or $y=2x$. Easy to plot!

(g) Recall that, at appropriate points, the gradient $\nabla f$ is orthogonal to any curve $f(x, y) = c$. In particular, $\nabla f\Big|_{(1,2)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is orthogonal to $f(x,y) = 1$ at $(1,2)$.

[The point $(1,2)$ lies on $y = 2x$, which is the relevant part of the level curve $f(x,y) = 1$ as determined in the previous part. The line $y = 2x$ has direction vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (make sure that this is obvious!), which is indeed orthogonal to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.]

(h) The derivative of $f(x,y)$ at $(1,2)$ in direction $\boldsymbol{v} = 3\boldsymbol{i} + \boldsymbol{j}$ is

$$\nabla f\Big|_{(1,2)} \cdot \frac{3\boldsymbol{i}+\boldsymbol{j}}{\|3\boldsymbol{i}+\boldsymbol{j}\|} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{9+1}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{-6+1}{\sqrt{10}} = -\frac{5}{\sqrt{10}}.$$

(i) As above, $\nabla f\Big|_{(1/2,2)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is orthogonal to $f(x,y) = 2$ at $(1/2, 2)$.

[Note that $f(1/2,2) = 2$, so that the point $(1/2,2)$ indeed lies on the level curve $f(x,y) = 2$.]

(j) $f(x,y)$ at $(1/2,2)$ increases most rapidly in the direction $\nabla f\Big|_{(1/2,2)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

[On the other hand, $f(x,y)$ at $(1/2,2)$ has zero derivative in the direction orthogonal to $\nabla f\Big|_{(1/2,2)}$.]

(k) The linearization of $f(x, y)$ at $(1, 2)$ is $L(x, y) = 1 - 2(x-1) + 1(y-2)$. Hence, the graph $z = f(x, y)$ is approximated by the tangent plane $z = L(x, y)$, that is, $z = 1 - 2(x-1) + 1(y-2)$. This simplifies to $2x - y + z = 1$.

*Alternative solution.* We are talking about the surface $z = f(x, y)$ at the point $(1, 2, 1)$ or, equivalently, $g(x, y, z) = 0$ with $g(x, y, z) = f(x, y) - z$. We compute $\nabla g\Big|_{(1,2,1)} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ which is a normal vector for the tangent plane. This plane thus is of the form $-2x + y - z = d$, and we find $d = -1$ using the point $(1,2,1)$.

(l) Using the chain rule, we have

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y-4x}{(1+2x^2-xy)^2}\cdot 1 + \frac{x}{(1+2x^2-xy)^2}(-\sin(t)) = \frac{y-4x-x\sin(t)}{(1+2x^2-xy)^2} \\ &= \frac{\cos(t) - 4(2+t) - (2+t)\sin(t)}{(1+2(2+t)^2-(2+t)\cos(t))^2}. \end{aligned}$$

We get the same answer by computing

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{1+2(2+t)^2-(2+t)\cos(t)} = \ldots = \frac{\cos(t)-4(2+t)-(2+t)\sin(t)}{(1+2(2+t)^2-(2+t)\cos(t))^2}.$$

(m) $\nabla f = \frac{1}{(1+2x^2-xy)^2}\begin{bmatrix} y-4x \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only leads to $x=0,\ y=0$. The only critical point is $(0,0)$.

A bit of computation shows that $f_{xx}(0,0)=-4,\ f_{yy}(0,0)=0,\ f_{xy}(0,0)=1$. Hence, the Hessian is $f_{xx}f_{yy}-f_{xy}^2 = -1$. This shows that $(0,0)$ is a saddle point.

*Note.* On the other hand, observe that $(0,0)$ is part of the level curve $f(x,y)=1$. Can you visualize what is happening?

□

**Problem 2.**

(a) Find all local extreme values and saddle points of the function $f(x,y)=\ln(x+y)+x^2-y$.

(b) Find all local extreme values and saddle points of the function $f(x,y)=x+2x^2+x^3+xy+y^2$.

**Solution.**

(a) To find the critical points, we need to solve the two equations $f_x = \frac{1}{x+y}+2x$ and $f_y = \frac{1}{x+y}-1=0$ for the two unknowns $x,y$. The second equation simplifies to $x+y=1$. Using this in the first equation gives $1+2x=0$, so that $x=-\frac{1}{2}$. From $x+y=1$, it follows that $y=\frac{3}{2}$.

The only critical point therefore is $\left(-\frac{1}{2},\frac{3}{2}\right)$.

$f_{xx} = -\frac{1}{(x+y)^2}+2,\ f_{yy} = -\frac{1}{(x+y)^2},\ f_{xy} = -\frac{1}{(x+y)^2}$

$f_{xx}f_{yy}-f_{xy}^2\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)} = 1\cdot(-1)-(-1)^2=-2<0$. Hence, $\left(-\frac{1}{2},\frac{3}{2}\right)$ is a saddle point.

(b) To find the critical points, we need to solve the two equations $f_x = 1+4x+3x^2+y=0$ and $f_y = x+2y=0$ for the two unknowns $x,y$.

Here, the second equation simplifies to $y=-\frac{x}{2}$. Substituting that in the first equation, we get $1+\frac{7}{2}x+3x^2=0$. This is a quadratic equation with solutions $x=-\frac{1}{2}$ and $x=-\frac{2}{3}$.

If $x=-\frac{1}{2}$ then $y=-\frac{x}{2}=\frac{1}{4}$, and we get the point $\left(-\frac{1}{2},\frac{1}{4}\right)$. If $x=-\frac{2}{3}$ then $y=-\frac{x}{2}=\frac{1}{3}$, and we get the point $\left(-\frac{2}{3},\frac{1}{3}\right)$.

In conclusion, the critical points are $\left(-\frac{1}{2},\frac{1}{4}\right)$, $\left(-\frac{2}{3},\frac{1}{3}\right)$.

$\left[f_{xx}f_{yy}-f_{xy}^2\right]_{\left(-\frac{1}{2},\frac{1}{4}\right)} = 1>0$ and $f_{xx}=1>0$. Hence, $\left(-\frac{1}{2},\frac{1}{4}\right)$ is a local minimum.

$\left[f_{xx}f_{yy}-f_{xy}^2\right]_{\left(-\frac{2}{3},\frac{1}{3}\right)} = -1<0$. Hence, $\left(-\frac{2}{3},\frac{1}{3}\right)$ is a saddle point. □

Armin Straub
straub@southalabama.edu

**Problem 3.**

(a) Let $g(x)$ be a function of one variable such that $g'(x) = e^{x^2}$. Let $w = g(st + e^t)$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

(b) Let $f(x,y)$ be some function of two variables. Write down a chain rule for $\frac{\partial}{\partial w} f(x(u,v,w), y(u,v,w))$.

(c) Write down a chain rule for $\frac{\partial}{\partial r} f$ and $\frac{\partial}{\partial \theta} f$ for $f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$.

(d) **(Challenge!)** Write down a chain rule for $\frac{\partial^2}{\partial r^2} f$ for $f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$.

**Solution.**

(a) Here, $w = g(x)$ with $x = st + e^t$. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} = e^{x^2} t = t e^{(st+e^t)^2}$, $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} = e^{x^2}(s + e^t) = (s + e^t) e^{(st+e^t)^2}$.

(b) $\frac{\partial}{\partial w} f(x(u,v,w), y(u,v,w)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w}$

(c) $\frac{\partial}{\partial r} f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$, $\frac{\partial}{\partial \theta} f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$

(d) $\frac{\partial^2}{\partial r^2} f = \frac{\partial}{\partial r}[f_x\cos\theta + f_y\sin\theta] = \left[\frac{\partial}{\partial r} f_x\right]\cos\theta + \left[\frac{\partial}{\partial r} f_y\right]\sin\theta$
$= [f_{xx}\cos\theta + f_{xy}\sin\theta]\cos\theta + [f_{yx}\cos\theta + f_{yy}\sin\theta]\sin\theta = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$

[Note that we used our previous result for $\frac{\partial}{\partial r} f$, which holds for any function $f$, to determine $\frac{\partial}{\partial r} f_x$ and $\frac{\partial}{\partial r} f_y$.] □

**Problem 4.** Consider the function $f(x,y,z) = xyz^2 + 4\sqrt{3 + yz}$.

(a) Compute the gradient $\nabla f$.

(b) Find the linearization of $f(x,y,z)$ at $(2,1,1)$.

(c) Find the derivative of $f(x,y,z)$ at $(2,1,1)$ in direction $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

(d) Compute the partial derivative $f_{zyx}$.

(e) Determine a normal vector for the surface $f(x,y,z) = 7$ at $(-1,1,1)$.

(f) Find equations for the tangent plane and normal line for the surface $f(x,y,z) = 7$ at the point $(-1,1,1)$.

(g) Find the line tangent to the curve of intersection of the surfaces $x^2yz = 1$ and $f(x,y,z) = 7$ at the point $(-1,1,1)$.

**Solution.**

(a) $\nabla f = \begin{bmatrix} yz^2 \\ xz^2 + \frac{2z}{\sqrt{3+yz}} \\ 2xyz + \frac{2y}{\sqrt{3+yz}} \end{bmatrix}$

(b) We have $f(2,1,1) = 10$ and $\nabla f\Big|_{(2,1,1)} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

Hence, the linearization of $f(x,y,z)$ at $(2,1,1)$ is $L(x,y,z)=10+1(x-2)+3(y-1)+5(z-1)$.

(c) The derivative of $f(x,y,z)$ at $(2,1,1)$ in direction $\boldsymbol{v}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ is

$$\nabla f\Big|_{(2,1,1)}\cdot\frac{\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}}{\|\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}\|}=\begin{bmatrix}1\\3\\5\end{bmatrix}\cdot\frac{1}{\sqrt{1+1+1}}\begin{bmatrix}1\\1\\-1\end{bmatrix}=\frac{1+3-5}{\sqrt{3}}=-\frac{1}{\sqrt{3}}.$$

(d) Since the order doesn't matter, we compute $f_{zyx}=f_{xyz}=\frac{\partial}{\partial z}\frac{\partial}{\partial y}yz^2=\frac{\partial}{\partial z}z^2=2z$.

(e) Recall that, at appropriate points, the gradient $\nabla f$ is orthogonal to any surface $f(x,y,z)=c$. In particular, $\nabla f\Big|_{(-1,1,1)}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ is orthogonal to $f(x,y,z)=7$ at $(-1,1,1)$.

[Note that $f(-1,1,1)=7$, so that the point $(-1,1,1)$ indeed lies on the level surface $f(x,y,z)=7$.]

(f) $\nabla f\Big|_{(-1,1,1)}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ is a normal vector for the tangent plane, which is therefore of the form $x-z=d$, and we find $d=-1-1=-2$ using the point $(-1,1,1)$. The tangent plane is $x-z=-2$.

The normal line has the parametrization $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\1\\1\end{bmatrix}+t\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}-1+t\\1\\1-t\end{bmatrix}$.

(g) If we work very close to the point $(-1,1,1)$, then the two surfaces are very nearly two planes. The line tangent to the curve of intersection is therefore just the line of intersection of these two planes (and we have learned earlier how to intersect two planes).

We already know that the surface $f(x,y,z)=7$ at $(-1,1,1)$ has normal vector $\nabla f\Big|_{(-1,1,1)}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

The surface $g(x,y,z)=x^2yz=1$ at $(-1,1,1)$ has normal vector $\nabla g\Big|_{(-1,1,1)}=(2xyz,x^2z,x^2y)\Big|_{(-1,1,1)}=(-2,1,1)$.

The line we are looking for lies in both of the corresponding tangent planes, and hence is orthogonal to both normal vectors. We can thus find its direction via the cross product:

$$\begin{bmatrix}1\\0\\-1\end{bmatrix}\times\begin{bmatrix}-2\\1\\1\end{bmatrix}=\begin{bmatrix}0-(-1)\\2-1\\1-0\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$ The line therefore is $$\begin{bmatrix}-1\\1\\1\end{bmatrix}+t\begin{bmatrix}1\\1\\1\end{bmatrix}.$$ □

**Problem 5.**

(a) Minimize $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $g(x,y,z)=x^2-z^2-1=0$.

In other words, find the point(s) on (the hyperbolic cylinder) $x^2-z^2-1=0$ that are closest to the origin.

(b) Determine a system of equations for finding the extreme values of $f(x,\,y,\,z)=x-y+2z$ on the sphere $x^2+y^2+z^2=3$.

In this case, it is actually not hard to solve that system. You will find two candidates for extrema. For geometric reasons, one of these has to be a maximum and the other a minimum. (Can you explain why that has to be the case?)

**Solution.**

(a) At such a minimizing point, we should have that $\nabla f$ and $\nabla g$ point in the same direction!

Why? The derivative of $f$ in directions $\boldsymbol{u}$ allowed by the constraint should be zero. Since these derivatives are $\nabla f \cdot \boldsymbol{u}$, this means that $\nabla f$ should be orthogonal to $g(x,y,z)=0$. That in turn means that $\nabla f$ and $\nabla g$ point in the same direction. (This is a quick rundown of the reasoning behind the method of Lagrange multipliers. Can you follow it?)

$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, $\nabla g = \begin{bmatrix} 2x \\ 0 \\ -2z \end{bmatrix}$. We need to solve $\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 0 \\ -2z \end{bmatrix}$, $x^2 - z^2 - 1 = 0$.

These are four equations, namely

$$\begin{aligned} 2x &= 2\lambda x \\ 2y &= 0 \\ 2z &= -2\lambda z \\ x^2 - z^2 - 1 &= 0, \end{aligned}$$

which need to be solved for the four unknowns.

Obviously, $y=0$. Note that $x \neq 0$ because otherwise $x^2 - z^2 - 1 = 0$ has no solution. Therefore, we can divide the first equation by $x$ to find $\lambda = 1$. Using the third equation, we then get $z=0$. Finally, the fourth equation then gives $x^2 - 1 = 0$ so that $x = \pm 1$.

We therefore have the two critical points $(\pm 1, 0, 0)$.

We note that $f(\pm 1, 0, 0) = 1$ for both points. For geometric reasons, these are therefore both the desired minima.

(b) Let $g(x,y,z) = x^2 + y^2 + z^2$. By the method of Lagrange multipliers, we need to find values $x, y, z, \lambda$ such that

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 3.$$

Since $\nabla f = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, these equations become:

$$\begin{aligned} 1 &= 2\lambda x \\ -1 &= 2\lambda y \\ 2 &= 2\lambda z \\ x^2 + y^2 + z^2 &= 3 \end{aligned}$$

These are four equations and four unknowns. Solving this system, we expect to get a handful of individual solutions. These then are the candidates for local extrema.

*Extra thoughts.* Because the sphere is closed and bounded (that's called compact), the function $f$ has to have both a maximum and a minimum value. We will now solve the system to find these.

One strategy to solve is to express $x$, $y$, $z$ in terms of $\lambda$ and use the values in the last equation: $x = \frac{1}{2\lambda}$, $y = -\frac{1}{2\lambda}$, $z = \frac{1}{\lambda}$ substituted in $x^2 + y^2 + z^2 = 3$ produces $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 3$. This simplifies to $\frac{1}{4} + \frac{1}{4} + 1 = 3\lambda^2$, that is, $\lambda^2 = \frac{1}{2}$, which has the two solutions $\lambda = \pm \frac{1}{\sqrt{2}}$.

In the case $\lambda = \frac{1}{\sqrt{2}}$, we get $x = \frac{1}{2\sqrt{2}}$, $y = -\frac{1}{2\sqrt{2}}$, $z = \frac{1}{\sqrt{2}}$. The value of $f$ at that point is $f\left(\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$.

In the case $\lambda = -\frac{1}{\sqrt{2}}$, we get $x = -\frac{1}{2\sqrt{2}}$, $y = \frac{1}{2\sqrt{2}}$, $z = -\frac{1}{\sqrt{2}}$. The value of $f$ at that point is $f\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{3}{\sqrt{2}}$.

Hence, $f\left(\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} \approx 2.121$ is the maximum and $f\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{3}{\sqrt{2}} \approx -2.121$ the minimum.

[For instance, the point $(1,1,1)$ is also on that sphere and, indeed, $f(1,1,1) = 2 < 2.121$.] □

**Problem 6.** Consider the iterated integral $\int_0^4 \int_{2-x/2}^{\sqrt{4-x}} x y \mathrm{d}y \mathrm{d}x$.

(a) Evaluate the integral.

(b) Interchange the order of integration.

(If you have time, evaluate that second integral and verify that it gives the same value.)

**Solution.**

(a) $\int_0^4 \int_{2-x/2}^{\sqrt{4-x}} x y \mathrm{d}y \mathrm{d}x = \int_0^4 \left[\frac{xy^2}{2}\right]_{y=2-x/2}^{y=\sqrt{4-x}} \mathrm{d}y \mathrm{d}x = \int_0^4 \left(\frac{x(4-x)}{2} - \frac{x(2-x/2)^2}{2}\right) \mathrm{d}x$
$= \int_0^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) \mathrm{d}x = \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_{x=0}^{x=4} = \frac{8}{3}$

(b) Interchanging the order of integration leads to $\int_0^2 \int_{4-2y}^{4-y^2} x y \mathrm{d}x \mathrm{d}y$.

*Evaluating that second integral.* $\int_0^2 \int_{4-2y}^{4-y^2} x y \mathrm{d}x \mathrm{d}y = \int_0^2 \left[\frac{x^2 y}{2}\right]_{x=4-2y}^{x=4-y^2} \mathrm{d}y = \int_0^2 \left(\frac{(4-y^2)^2 y}{2} - \frac{(4-2y)^2 y}{2}\right) \mathrm{d}y$
$= \frac{1}{2}\int_0^2 (16y^2 - 12y^3 + y^5) \mathrm{d}y = \frac{1}{2}\left[\frac{16y^3}{3} - 3y^4 + \frac{y^6}{6}\right]_{y=0}^{y=2} = \frac{8}{3}$

□

**Problem 7.** Consider the region $R$ with $x^2 + y^2 \leqslant 4$ and $y \geqslant 0$. Write down an iterated integral for the area of $R$

(a) using vertical cross-sections,

(b) using horizontal cross-sections,

(c) using polar coordinates.

**Solution.**

(a) $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \mathrm{d}y \mathrm{d}x$

(b) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \mathrm{d}x \mathrm{d}y$

(c) $\int_0^{\pi} \int_0^2 r \mathrm{d}r \mathrm{d}\theta$ (Don't forget the $r$ in the integrand!) □

**Problem 8.** Convert the cartesian integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{1+x^2+y^2} \mathrm{d}y \mathrm{d}x$ into an equivalent polar integral. Then evaluate the polar integral.

**Solution.** $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{1+x^2+y^2} \mathrm{d}y \mathrm{d}x = \int_0^{\pi/2} \int_0^2 \frac{1}{1+r^2} r \mathrm{d}r \mathrm{d}\theta = \int_0^{\pi/2} \left[\frac{\ln(1+r^2)}{2}\right]_{r=0}^{r=2} \mathrm{d}\theta$
$= \int_0^{\pi/2} \frac{\ln(5)}{2} \mathrm{d}\theta = \frac{\pi \ln(5)}{4}$

*Note.* It would be a nightmare to try and evaluate the original integral without converting to polar coordinates. □

Armin Straub
straub@southalabama.edu