Example 98. (newsflash, 12/2018!) A new largest (proven) prime was found: $2^{82,589,933}-1$.

https://www.mersenne.org/primes/?press=M82589933
This is a **Mersenne prime** (like the last 17 record primes). It has a bit over 24.8 million (decimal) digits (versus 23.2 for the previous record). The prime was found as part of GIMPS (Great Internet Mersenne Prime Search), which offers a $3,000 award for each new Mersenne prime discovered.

The EFF (Electronic Frontier Foundation) is offering $150,000 (donated anonymously for that specific purpose) for the discovery of the first prime with at least 100 million decimal digits.

https://www.eff.org/awards/coop
[Prizes of $50,000 and $100,000 for primes with 1 and 10 million digits have been claimed in 2000 and 2009.]

Example 99. (geometric sum) Evaluate $1+x+x^2+\ldots+x^n$.

Solution. $(1+x+x^2+\ldots+x^n)(x-1)=x^{n+1}-1$, so that $1+x+x^2+\ldots+x^n=\frac{x^{n+1}-1}{x-1}$.

Geometric series. In particular, $\sum_{k=1}^{\infty} x^k = \lim_{n\to\infty} \frac{x^{n+1}-1}{x-1} = \frac{1}{1-x}$, provided that $|x|<1$.

Lemma 100. If $r\mid n$, then $x^r-1\mid x^n-1$.

Proof. Write $n=rs$. It follows from $x^s-1=(x-1)(1+x+x^2+\ldots+x^{s-1})$ that $x^{rs}-1=(x^r-1)(1+x^r+x^{2r}+\ldots+x^{r(s-1)})$. □

Definition 101. A **Mersenne prime** is a prime of the form 2^n-1.

For instance. The first few Mersenne primes have exponents $2,3,5,7,13,17,19,31,61,89,107,\ldots$ All of these exponents are primes (but not all primes work: for instance, $2^{11}-1=23\cdot 89$). See next result.

Anecdote. Euler proved in 1772 that $2^{31}-1$ is prime (then, and until 1867, the largest known prime).
"$2^{31}-1$ is probably the greatest [Mersenne prime] that ever will be discovered; for as they are merely curious, without being useful, it is not likely that any person will attempt to find one beyond it." — P. Barlow, 1811
https://en.wikipedia.org/wiki/2,147,483,647
Mersenne primes give rise precisely to all even perfect numbers (numbers whose proper divisors sum to the number itself; for instance, 6 is perfect because $6=1+2+3$). Indeed, Euclid showed that, if 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is perfect [$p=2$: $2\cdot 3=6$, $p=3$: $4\cdot 7=28=1+2+4+7+14$, $p=5$: $16\cdot 33=528$, ...]. It is not known whether odd perfect numbers exist.

Corollary 102. 2^n-1 can only be prime if n is prime.

Proof. It follows from the previous lemma that, if $n=rs$ is composite, then 2^n-1 is divisible by 2^r-1 (as well as 2^s-1). □

For instance. $2^6-1=63$ is divisible by both $2^2-1=3$ and $2^3-1=7$.

Example 103. (combinatorial warm-up) A typical Amazon gift card code is of the form

$$\text{6DAG-KJ2PZ5-3ATM.}$$

Suppose that, at any time, say, one million gift cards are active. What are the odds that a random gift card code is a valid one? Is this a security issue?

Solution. It seems that each of the $4+6+4=14$ letters is either a capital letter or a digit, meaning there are $26+10=36$ possibilities for each (though, in actuality, for instance, a letter like O might not be used because of possible confusion with 0). In total, there are $36^{14} \approx 6.14 \cdot 10^{21}$ many possible such codes.

If 10^6 codes are valid, then the odds are $10^6 / (6.14 \cdot 10^{21}) \approx 1.63 \cdot 10^{-16}$.

If you were able to go through one million random codes per second, then it would still take about 195 years on average until you ran into a valid code.

Variation. As part of a settlement, I received a Vitamix 70 USD gift code much like Q4VTA72SNV3. How does that change the odds? Just as above, but with 11 instead of 14 letters. Hence, the number of codes is smaller by a factor of $36^3 = 46,656$. (Still a lot of codes; but, if one million gift cards are active [almost certainly a lot less, so adjust accordingly] and if you were able to go through one million random codes per second, you would now expect success in about 1.5 days.)

Example 104. (bonus challenge!) Find the smallest (pseudo)prime with 100 decimal digits, all of which are odd.

(Send me an email by Feb 24 with the prime, and how you found it, to collect a bonus point. Earn an extra bonus point if you can find it using a single line of Sage code [artificial concatenations not allowed].)