

- Recall that, in contrast to DES, the operations of AES have very simple (though somewhat advanced) mathematical descriptions.

No mysteriously constructed S-boxes and P-boxes as in DES.

ByteSub (continued)

Each of the 16 bytes gets substituted as follows.

- Interpret the input byte $(b_7b_6\dots b_0)_2$ as the element $b_7x^7 + \dots + b_1x + b_0$ of $\text{GF}(2^8)$.
- Compute $s^{-1} = c_0 + c_1x + \dots + c_7x^7$ (with 0^{-1} interpreted as 0).

Important comment. This inversion is what makes AES highly nonlinear.

If the ByteSub substitution was linear, then all of AES would be linear (because all other layers are linear; assuming we adjust the key schedule accordingly).

- Then the output bits $(d_7d_6\dots d_1d_0)_2$ are

$$\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_7 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Comment. The particular choice of matrix and vector has the effect that no ByteSub output equals the ByteSub input (or its complement).

Example 134. Invert $x^3 + 1$ in $\text{GF}(2^8)$, constructed as in AES. [Example 131, again]

Solution. We use the extended Euclidean algorithm, and always reduce modulo 2:

$$\begin{aligned} x^8 + x^4 + x^3 + x + 1 &\equiv (x^5 + x^2 + x + 1) \cdot x^3 + 1 \\ x^3 + 1 &\equiv x \cdot x^2 + 1 \end{aligned}$$

Backtracking through this, we find that Bézout's identity takes the form

$$\begin{aligned} 1 &\equiv 1 \cdot x^3 + 1 - x \cdot x^2 \\ &\equiv (x^6 + x^3 + x^2 + x + 1) \cdot x^3 + 1 + x \cdot (x^8 + x^4 + x^3 + x + 1) - (x^5 + x^2 + x + 1) \cdot x^3 \end{aligned}$$

Hence, $(x^3 + 1)^{-1} = x^6 + x^3 + x^2 + x + 1$ in $\text{GF}(2^8)$.

Example 135. (homework)

- What happens to the byte $(0000\ 0101)_2$ during ByteSub?
- What happens to the byte $(0000\ 1001)_2$ during ByteSub?

Solution.

(a) $(0000\ 0101)_2$ represents the polynomial $x^2 + 1$.

By the previous example, its inverse is $(x^2 + 1)^{-1} = x^6 + x^4 + x$ in $\text{GF}(2^8)$, which is $c = (0101\ 0010)_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

[This is just the usual matrix-vector product modulo 2. The highlighted columns are the ones which get added up during this matrix-vector product.]

Hence, the output of ByteSub is the byte $(0110\ 1011)_2$.

Comment. To check, this indeed matches the value $107 = (0110\ 1011)_2$ in the lookup table given in Table 5.1 of our book (row 0, column $(0101)_2 = 5$).

(b) $(0000\ 1001)_2$ represents the polynomial $x^3 + 1$.

By the previous example, $(x^3 + 1)^{-1} = x^6 + x^3 + x^2 + x + 1$ in $\text{GF}(2^8)$, which is $c = (0100\ 1111)_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the output of ByteSub is the byte $(0000\ 0001)_2$.

Comment. To check, this indeed matches the value 1 in the lookup table given in Table 5.1 of our book (row 0, column $(1001)_2 = 9$).

Review: multiplicative order and primitive roots

Definition 136. The **multiplicative order** of an invertible residue a modulo n is the smallest positive integer k such that $a^k \equiv 1 \pmod{n}$.

Important note. By Euler's theorem, the multiplicative order can be at most $\phi(n)$.

Example 137. What is the multiplicative order of $2 \pmod{7}$?

Solution. $2^1 = 2$, $2^2 = 4$, $2^3 \equiv 1 \pmod{7}$. Hence, the multiplicative order of $2 \pmod{7}$ is 3.

Definition 138. If the multiplicative order of an residue a modulo n equals $\phi(n)$ [in other words, the order is as large as possible], then a is said to be **primitive root** modulo n .

A primitive root is also referred to as a **multiplicative generator** (because the products of a and itself, that is, $1, a, a^2, a^3, \dots$, produce all invertible residues).

Example 139. What is the multiplicative order of $3 \pmod{7}$?

Solution. $3^1 = 3$, $3^2 \equiv 2$, $3^3 \equiv 6$, $3^4 \equiv 4$, $3^5 \equiv 5$, $3^6 \equiv 1$. Hence, the multiplicative order of $3 \pmod{7}$ is 6. This means that 3 is a primitive root modulo 7. Note how every (invertible) residue shows up as a power of 3.