Example 77. (review) The solutions to $x^2 \equiv 9 \pmod{35}$ are ± 3 and $\pm 17 \pmod{35}$.

Example 78. Determine all solutions to $x^2 \equiv 4 \pmod{105}$.

Solution. By the CRT:

$$\begin{aligned} & x^2 \equiv 4 \pmod{105} \\ \Longleftrightarrow\ & x^2 \equiv 4 \pmod 3 \text{ and } x^2 \equiv 4 \pmod 5 \text{ and } x^2 \equiv 4 \pmod 7 \\ \Longleftrightarrow\ & x \equiv \pm 2 \pmod 3 \text{ and } x \equiv \pm 2 \pmod 5 \text{ and } x \equiv \pm 2 \pmod 7 \end{aligned}$$

At this point, we see that there are $2^3 = 8$ solutions.

For instance, let us find the solution corresponding to $x \equiv 2 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv -2 \pmod 7$:

$$x \equiv 2 \cdot 5 \cdot 7 \cdot \underbrace{[(5 \cdot 7)^{-1}_{\bmod 3}]}_{-1} + 2 \cdot 3 \cdot 7 \cdot \underbrace{[(3 \cdot 7)^{-1}_{\bmod 5}]}_{1} - 2 \cdot 3 \cdot 5 \cdot \underbrace{[(3 \cdot 5)^{-1}_{\bmod 7}]}_{1} \equiv -70 + 42 - 30 = -58 \equiv 47$$

Similarly, we find all eight solutions (note how the solutions pair up):

(mod 3)	(mod 5)	(mod 7)	(mod 105)
2	2	2	2
−2	−2	−2	−2
2	2	−2	47
−2	−2	2	−47
2	−2	2	23
−2	2	−2	−23
−2	2	2	37
2	−2	−2	−37

The complete list of solutions is: $\pm 2, \pm 23, \pm 37, \pm 47$

Silicon slave labor. Once we are comfortable doing it by hand, we can easily let Sage do the work for us:

```
Sage] crt([2,2,-2], [3,5,7])
```

47

```
Sage] solve_mod(x^2 == 4, 105)
```

$[(37), (82), (58), (103), (2), (47), (23), (68)]$

Review: quadratic residues

Definition 79. An integer a is a **quadratic residue** modulo n if $a \equiv x^2 \pmod n$ for some x.

Important note. Products of quadratic residues are quadratic residues.

Example 80. List all quadratic residues modulo 11.

Solution. We compute all squares: $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 9$, $(\pm 4)^2 \equiv 5$, $(\pm 5)^2 \equiv 3$. Hence, the quadratic residues modulo 11 are $0, 1, 3, 4, 5, 9$.

Important comment. Exactly half of the 10 nonzero residues are quadratic. Can you explain why?

[*Hint.* $x^2 \equiv y^2 \pmod p \iff (x - y)(x + y) \equiv 0 \pmod p \iff x \equiv y$ or $x \equiv -y \pmod p$]

Example 81. List all quadratic residues modulo 15.

Solution. We compute all squares modulo 15: $0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 9$, $(\pm 4)^2 \equiv 1$, $(\pm 5)^2 \equiv 10$, $(\pm 6)^2 \equiv 6$, $(\pm 7)^2 \equiv 4$. Hence, the quadratic residues modulo 15 are $0, 1, 4, 6, 9, 10$.

Important comment. Among the $\phi(15) = 8$ invertible residues, the quadratic ones are $1, 4$ (exactly a quarter). Note that 15 is of the form $n = pq$ with p, q distinct primes.

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Theorem 82. Let p, q, r be distinct odd primes.

- The number of invertible residues modulo n is $\phi(n)$.
- The number of invertible quadratic residues modulo p is $\frac{\phi(p)}{2} = \frac{p-1}{2}$.
- The number of invertible quadratic residues modulo pq is $\frac{\phi(pq)}{4} = \frac{p-1}{2}\frac{q-1}{2}$.
- The number of invertible quadratic residues modulo pqr is $\frac{\phi(pqr)}{8} = \frac{p-1}{2}\frac{q-1}{2}\frac{r-1}{2}$.
- $\ldots$

Proof.

- We already knew that the number of invertible residues modulo n is $\phi(n)$.
- Think about squaring all residues modulo p to make a complete list of all quadratic residues. Let a^2 be one of the nonzero quadratic residues. As we observed earlier, $x^2 \equiv a^2 \pmod{p}$ has exactly 2 solutions, meaning that exactly two residues (namely $\pm a$) square to a^2. Hence, the number of invertible quadratic residues modulo p is half the number of invertible residues modulo p.
- Again, think about squaring all residues modulo pq to make a complete list of all quadratic residues. Let a^2 be one of the invertible quadratic residues. By the CRT, $x^2 \equiv a^2 \pmod{pq}$ has exactly 4 solutions (why is it important that a is invertible here?!), meaning that exactly four residues square to a^2. Hence, the number of invertible quadratic residues modulo pq is a quarter of the number of invertible residues modulo pq.
- Spell out the situation modulo pqr! □

Comment. Make similar statements when one of the primes is equal to 2.

Example 83. (bonus!) What is the total number of quadratic residues modulo pqr if p, q, r are distinct odd primes? (To collect a bonus point, send me the answer and a short explanation by next week.)

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