ODEs vs PDEs

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 22. The DE

$$\left(\frac{\partial}{\partial x}\right)^2 u(x,y) + \left(\frac{\partial}{\partial y}\right)^2 u(x,y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for instance, steady-state heat distributions.
https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x,y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x,y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 23. (existence and uniqueness) Consider the IVP $y' = f(x,y)$, $y(a) = b$.
If both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous [in a rectangle] around (a,b), then the IVP has a unique solution in some interval $x \in (a-\delta, a+\delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y}f(x,y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

Example 24. Consider the IVP $(x-y^2)y' = 3x$, $y(4) = b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

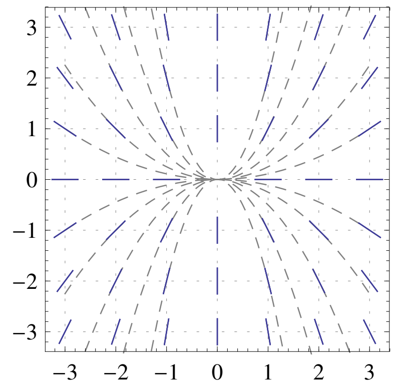
Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = 3x/(x-y^2)$. We compute that $\frac{\partial}{\partial y}f(x,y) = 6xy/(x-y^2)^2$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $x-y^2 \neq 0$.

Note that $4-b^2 \neq 0$ is equivalent to $b \neq \pm 2$.

Hence, if $b \neq \pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 25. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = 2y/x$.

We compute that $\frac{\partial}{\partial y}f(x,y) = 2/x$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

What happens in the case $a = 0$?

Solution. In Example 18, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions.

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

Example 26. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = ky^2$. We compute that $\frac{\partial}{\partial y}f(x,y) = 2ky$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 27. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{dy}{dx} = k$. Integrating $\int \frac{1}{y^2}dy = \int k\,dx$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C+kx} = \frac{1}{D-kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 28. Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D-x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1-x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x) = \frac{1}{1-x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

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