Review. Underdamped, overdamped, critically damped for the DE $y'' + dy' + cy = 0$.

The critically damped case is when the damping is just large enough to avoid oscillations (underdamped case). If damping is increased further (overdamped case), then this typically means that it "takes longer" for the solution to approach equilibrium.

Challenge. Try to make this last comment more precise! For that, focus on the exponential term in the corresponding general solution that decays the slowest. That slowest term decays fastest in the critically damped case.

The same DE describes the current in an RLC circuit (disconnected from its source).

Example 101. The motion of a mass on a spring is described by $my'' + 3y' + 2y = 0$. For which value of m is the motion critically damped? Underdamped? Overdamped?

Solution. The characteristic roots are $\frac{1}{2}(-3 \pm \sqrt{9 - 8m})$. The motion is critically damped if $9 - 8m = 0$. Equivalently, the motion is critically damped $m = \frac{9}{8}$.

Consequently, the motion is underdamped if $m > \frac{9}{8}$ (then we get complex roots and the solutions will involve oscillations), and it is overdamped if $m < \frac{9}{8}$ (the roots are real and the solutions will not involve oscillations).

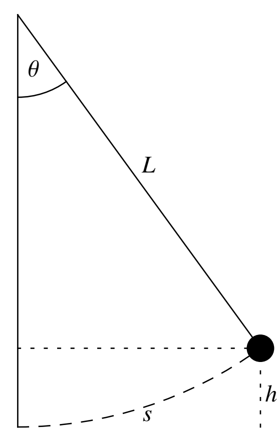
Application: motion of a pendulum

Example 102. Show that the motion of an ideal pendulum is described by

$$L\theta'' + g\sin(\theta) = 0,$$

where θ is the angular displacement and L is the length of the pendulum.

And, as usual, g is acceleration due to gravity.



For short times and small angles, this motion is approximately described by

$$L\theta'' + g\theta = 0.$$

This is because, if θ is small, then $\sin(\theta) \approx \theta$. For instance, for $\theta = 15°$ the error $\theta - \sin\theta$ is about 1%.

Solution. (Newton's second law) The tangential component of the gravitational force is $F = -\sin\theta \cdot mg$. Combining this with Newton's second law, according to which $F = ma = mL\theta''$ (note that $a = s''$ where $s = L\theta$), we obtain the claimed DE.

Solution. (conservation of energy) Alternatively, we can use conservation of energy to derive the DE. Again, we assume the string to be massless, and let m be the swinging mass. Let s and h be as in the sketch above.

The velocity (more accurately, the speed) of the mass is $v = \frac{\mathrm{d}s}{\mathrm{d}t} = L\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

Its kinetic energy therefore is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$.

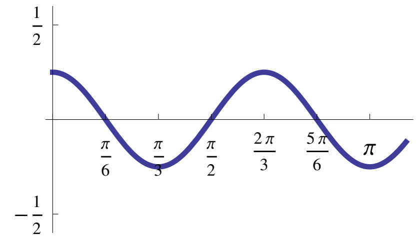
On the other hand, the potential energy is $V = mgh = mgL(1 - \cos\theta)$ (weight mg times height h).

By the principle of conservation of energy, the sum of these is constant: $T + V = \text{const}$

Taking the time derivative, this becomes $\frac{1}{2}mL^2 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgL\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Cancelling terms, we obtain the DE.

Example 103. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = 0$. What is the period and the amplitude of the resulting oscillations?

Solution. The roots of the characteristic polynomial are $\pm 3i$.
Hence, $\theta(t) = A\cos(3t) + B\sin(3t)$. $\theta(0) = A = 1/4$. $\theta'(0) = 3B = 0$.
Therefore, the solution is $\theta(t) = 1/4\cos(3t)$.
Hence, the period is $2\pi/3$ and the amplitude is $1/4$.



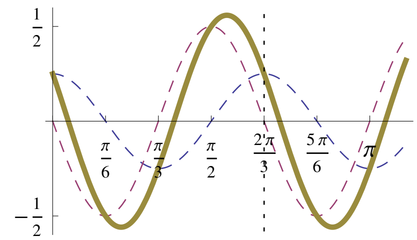
Comment. The initial angle $1/4$ is about $14.3°$.

Example 104. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = -\frac{3}{2}$ ("initial kick"). What is the period and the amplitude of the resulting oscillations?

Solution. This time, $\theta(0) = A = 1/4$. $\theta'(0) = 3B = -3/2$.

Therefore, the solution is $\theta(t) = \frac{1}{4}\cos(3t) - \frac{1}{2}\sin(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $\sqrt{\frac{1}{4^2} + \frac{1}{2^2}} = \frac{\sqrt{5}}{4} \approx 0.559$.



Comment. Using polar coordinates, we get $\theta(t) = \frac{\sqrt{5}}{4}\cos(3t - \alpha)$ with phase angle $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$.

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