Variation of constants for solving inhomogeneous linear DEs

Review. To find the general solution of an inhomogeneous linear DE $Ly = f(x)$, we only need to find a single **particular solution** y_p. Then the general solution is $y_p + y_h$, where y_h is the general solution of $Ly = 0$.

The **method of undetermined coefficients** allows us to find a particular solution to an inhomogeneous linear DE $Ly = f(x)$ for certain functions $f(x)$.

Moreover, the homogeneous DE needs to have constant coefficients.

The next method, known as **variation of constants** (or variation of parameters), has no restriction on the functions $f(x)$ (or the linear DE). The price to pay for this is that the method is usually more laborious.

Theorem 114. (variation of constants) A particular solution to the inhomogeneous second-order linear DE $Ly = y'' + P_1(x)y' + P_0(x)y = f(x)$ is given by:

$$y_p = C_1(x)y_1(x) + C_2(x)y_2(x), \quad C_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x, \quad C_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x,$$

where y_1, y_2 are independent solutions of $Ly = 0$ and $W = y_1y_2' - y_1'y_2$ is their Wronskian.

Comment. We obtain the general solution if we consider all possible constants of integration in the formula for y_p.

Proof. Let us look for a particular solution of the form $y_p = C_1(x)\,y_1(x) + C_2(x)\,y_2(x)$.
This "ansatz" is called **variation of constants/parameters**. We plug into the DE to determine conditions on C_1, C_2 so that y_p is a solution. The DE will give us one condition and (since there are two unknowns), it is reasonable to expect that we can impose a second condition (labelled below as "our wish") to make our life simpler.
We compute $y_p' = \underbrace{C_1'y_1 + C_2'y_2}_{=0\text{ (our wish)}} + C_1y_1' + C_2y_2'$ and, thus, $y_p'' = C_1'y_1' + C_2'y_2' + C_1y_1'' + C_2y_2''$.

["Our wish" was chosen so that y_p'' would only involve first derivatives of C_1 and C_2.]

Therefore, plugging into the DE results in

$$Ly_p = C_1'y_1' + C_2'y_2' + \underbrace{C_1y_1'' + C_2y_2'' + P_1(x)(C_1y_1' + C_2y_2') + P_0(x)(C_1y_1 + C_2y_2)}_{=C_1Ly_1 + C_2Ly_2 = 0} \overset{!}{=} f(x).$$

We conclude that y_p solves the DE if the following two conditions (the first is "our wish") are satisfied:

$$\begin{aligned} C_1'y_1 + C_2'y_2 &= 0, \\ C_1'y_1' + C_2'y_2' &= f(x). \end{aligned}$$

These are linear equations in C_1' and C_2'. Solving gives $C_1' = \frac{-y_2 f(x)}{y_1y_2' - y_1'y_2}$ and $C_2' = \frac{y_1 f(x)}{y_1y_2' - y_1'y_2}$, and it only remains to integrate. □

Comment. In matrix-vector form, the equations are $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Our solution then follows from multiplying $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{y_1y_2' - y_1'y_2}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$ with $\begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Advanced comment. $W = y_1y_2' - y_1'y_2$ is called the **Wronskian** of y_1 and y_2. In general, given a linear homogeneous DE of order n with solutions $y_1, \ldots, y_n$, the Wronskian of $y_1, \ldots, y_n$ is the determinant of the matrix where each column consists of the derivatives of one of the y_i. One useful property of the Wronskian is that it is nonzero if and only if the $y_1, \ldots, y_n$ are linearly independent and therefore generate the general solution.

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Example 115. Determine the general solution of $y'' - 2y' + y = \frac{e^x}{x}$.

Solution. This DE is of the form $Ly = f(x)$ with $L = D^2 - 2D + 1$ and $f(x) = \frac{e^x}{x}$.
Since $L = (D-1)^2$, the homogeneous DE has the two solutions $y_1 = e^x$, $y_2 = xe^x$.
The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^x(1+x)e^x - e^x(xe^x) = e^{2x}$.
By variation of parameters (Theorem 114), we find that a particular solution is

$$y_p = -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x = -e^x \int 1 \mathrm{d}x + xe^x \int \frac{1}{x} \mathrm{d}x = xe^x(\ln|x| - 1).$$

The general solution therefore is $xe^x(\ln|x| - 1) + (C_1 + C_2 x)e^x$.

If we prefer, a simplified particular solution is $xe^x \ln|x|$ (because we can add any multiple of xe^x to y_p). Then the general solution takes the simplified form $xe^x \ln|x| + (C_1 + C_2 x)e^x$.

Comment. Adding constants of integration in the formula for y_p, we get $-e^x(x + D_1) + xe^x(\ln|x| + D_2)$, which is the general solution. Any choice of constants suffices to give us a particular solution.

Important comment. Note that we cannot use the method of undetermined coefficients here because the inhomogeneous term $f(x) = \frac{e^x}{x}$ is not of the appropriate form. See the next example for a case where both methods can be applied.

Example 116. (homework) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

Solution.

(a) We already did this in Example 89: The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.
$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \stackrel{!}{=} e^{3x}$. Hence, $C = 1/25$.
Therefore, the general solution is $y(x) = \frac{1}{25} e^{3x} + (C_1 + C_2 x)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = e^{3x}$.
Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.
By variation of parameters (Theorem 114), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x \\ &= -e^{-2x} \underbrace{\int xe^{5x} \mathrm{d}x}_{= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}} + xe^{-2x} \underbrace{\int e^{5x} \mathrm{d}x}_{= \frac{1}{5}e^{5x}} = \frac{1}{25} e^{3x}. \end{aligned}$$

The general solution therefore is $\frac{1}{25} e^{3x} + (C_1 + C_2 x)e^{-2x}$, which matches what we got before.

Example 117. (homework) Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

Solution.

(a) We already did this in Example 90: The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.
$y_p' = C(-2x^2 + 2x)e^{-2x}$
$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$
$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$
It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = 7e^{-2x}$.
Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.
By variation of parameters (Theorem 114), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x \\ &= -e^{-2x} \underbrace{\int 7x \mathrm{d}x}_{=\frac{7}{2}x^2} + xe^{-2x} \underbrace{\int 7 \mathrm{d}x}_{=7x} = \frac{7}{2}x^2 e^{-2x}. \end{aligned}$$

The general solution therefore is $\frac{7}{2}x^2 e^{-2x} + (C_1 + C_2 x)e^{-2x}$, which matches what we got before.

Systems of differential equations

Modeling two connected fluid tanks

Example 118. Consider two brine tanks. Initially, tank T_1 is filled with 24gal water containing 3lb salt, and tank T_2 with 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

Derive a system of equations for the amount of salt in the tanks after t minutes.

Solution. Note that the amount of water in each tank is constant because the flows balance each other.
Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In the time interval $[t, t + \Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - (18 + 54) \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.
In conclusion, we have obtained the system of equations

$$\begin{aligned} y_1' &= -3y_1 + 2y_2 + 27, & y_1(0) &= 3, \\ y_2' &= 3y_1 - 8y_2, & y_2(0) &= 0. \end{aligned}$$

We will soon learn how to solve such systems of DEs.

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