Review. $\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$

Here, $u_a(t)f(t-a)$ is $f(t)$ delayed by a.

In particular. $\mathcal{L}(u_a(t)) = \frac{e^{-as}}{s}$ (here, we use $f(t)=1$ and $F(s)=\frac{1}{s}$).

Example 152. Determine the Laplace transform $\mathcal{L}(e^{rt}u_a(t))$.

Solution. Write $e^{rt}u_a(t) = f(t-a)u_a(t)$ with $f(t) = e^{r(t+a)} = e^{ra}e^{rt}$. Since $F(s) = \mathcal{L}(f(t)) = \frac{e^{ra}}{s-r}$, we have

$$\mathcal{L}(e^{rt}u_a(t)) = e^{-sa}F(s) = \frac{e^{-(s-r)a}}{s-r}.$$

Example 153. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right)$.

Solution. $\frac{1}{s+3}$ is the Laplace transform of e^{-3t}. Hence, $\frac{e^{-2s}}{s+3}$ is the Laplace transform of e^{-3t} delayed by 2. In other words, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right) = u_2(t)e^{-3(t-2)}$.

Comment. Note that this is one of the terms in our solution $Y(s)$ in Example 154 (because $s^2+5s+6 = (s+2)(s+3)$). Can you determine the full inverse Laplace transform of $Y(s)$?

In general. Likewise, we have $\mathcal{L}^{-1}\left(\frac{e^{-as}}{s-r}\right) = u_a(t)e^{r(t-a)}$ (namely, e^{rt} delayed by a).

Using these unit step functions, we can conveniently solve differential equations featuring certain kinds of discontinuities.

Note that the DE in our next example describes the motion of a mass on a spring with damping, where the external force is zero except for the time interval $[2,3)$ when we suddenly have a force equal to 5.

Example 154. Determine the Laplace transform of the unique solution to the initial value problem

$$y'' + 5y' + 6y = \begin{cases} 5, & \text{if } 2 \leqslant t < 3, \\ 0, & \text{otherwise,} \end{cases} \qquad y(0) = -4, \qquad y'(0) = 8.$$

Solution. First, we observe that the right-hand side of the differential equation can be written as $5(u_2(t) - u_3(t))$. It follows from the Laplace transform table that $\mathcal{L}(u_a(t)) = e^{-as}\frac{1}{s}$ (using the entry for $u_a(t)f(t-a)$ with $f(t)=1$). Consequently, $\mathcal{L}(5(u_2(t)-u_3(t))) = 5e^{-2s}\frac{1}{s} - 5e^{-3s}\frac{1}{s} = \frac{5}{s}(e^{-2s} - e^{-3s})$.

Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) = \frac{5}{s}(e^{-2s} - e^{-3s}),$$

which using the initial values simplifies to

$$(s^2+5s+6)Y(s) + 4s - 8 + 5\cdot 4 = \frac{5}{s}(e^{-2s} - e^{-3s}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2+5s+6}\left[\frac{5}{s}(e^{-2s} - e^{-3s}) - 4s - 12\right].$$

First challenge. Take the inverse Laplace transform to find $y(t)$! (See Examples 153 and 155.)

Second challenge. Solve the DE without using Laplace transforms! (First, solve the IVP for $t<2$ in which case we have no external force. That tells us what $y(2)$ and $y'(2)$ should be. Using these as the new initial conditions, solve the IVP for $t \in [2,3)$. Then, using $y(3)$ and $y'(3)$, solve the IVP for $t \geqslant 3$. In the end, you will have found the solution $y(t)$ in three pieces. On the other hand, the Laplace transform allows us to avoid working piece-by-piece.)

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Example 155. Solve the IVP $y'' + 3y' + 2y = f(t)$, $y(0) = y'(0) = 0$ with $f(t) = \begin{cases} 1, & 3 \leqslant t < 4, \\ 0, & \text{otherwise.} \end{cases}$

Solution. First, we write $f(t) = u_3(t) - u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2 Y(s) - s y(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} = (e^{-3s} - e^{-4s}) \frac{1}{s}.$$

Using that $s^2 + 3s + 2 = (s+1)(s+2)$, we find

$$Y(s) = (e^{-3s} - e^{-4s}) \frac{1}{s(s+1)(s+2)} = (e^{-3s} - e^{-4s}) \left[\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \right],$$

where A, B, C are determined by partial fractions (we compute the values below). Taking the inverse Laplace transform of each of the six terms in this product, as in Example 153, we find

$$y(t) = A(u_3(t) - u_4(t)) + B(u_3(t) e^{-(t-3)} - u_4(t) e^{-(t-4)}) + C(u_3(t) e^{-2(t-3)} - u_4(t) e^{-2(t-4)}).$$

If preferred, we can express this as $y(t) = \begin{cases} 0, & \text{if } t < 3, \\ A + B e^{-(t-3)} + C e^{-2(t-3)}, & \text{if } 3 \leqslant t < 4, \\ B(e^{-(t-3)} - e^{-(t-4)}) + C(e^{-2(t-3)} - e^{-2(t-4)}), & \text{if } t \geqslant 4. \end{cases}$

Finally, $A = \frac{1}{(s+1)(s+2)}\Big|_{s=0} = \frac{1}{2}$, $B = \frac{1}{s(s+2)}\Big|_{s=-1} = -1$, $C = \frac{1}{s(s+1)}\Big|_{s=-2} = \frac{1}{2}$.

Comment. Check that these values make $y(t)$ a continuous function (as it should be for physical reasons).

Example 156. (extra practice) Determine the Laplace transform of the unique solution to the initial value problem

$$y'' - 6y' + 5y = \begin{cases} 3e^{-2t}, & \text{if } 1 \leqslant t < 4, \\ 0, & \text{otherwise,} \end{cases} \qquad y(0) = 2, \quad y'(0) = -1.$$

Solution. First, we write the right-hand side of the differential equation as $f(t) := 3e^{-2t}(u_1(t) - u_4(t))$. By Example 152, the Laplace transform of $f(t)$ is $\mathcal{L}(f(t)) = 3\frac{e^{-(s+2)}}{s+2} - 3\frac{e^{-4(s+2)}}{s+2} = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)})$. Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2 Y(s) - s y(0) - y'(0) - 6(sY(s) - y(0)) + 5Y(s) = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}),$$

which using the initial values simplifies to

$$(s^2 - 6s + 5) Y(s) - 2s + 13 = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2 - 6s + 5} \left[\frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}) + 2s - 13 \right].$$