

# Midterm #2 – Practice

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**Reminder.** No notes, calculators or tools of any kind will be permitted on the midterm exam.

**Problem 1.** Let  $L$  be a linear differential operator of order 4 with constant real coefficients. Suppose that  $3 + 7i$  is a repeated characteristic root of  $L$ .

- (a) What is the general solution to  $Ly = 0$ ?
- (b) Write down the simplest form of a particular solution  $y_p$  of the DE  $Ly = 7x^2e^{3x}$  with undetermined coefficients.
- (c) Write down the simplest form of a particular solution  $y_p$  of the DE  $Ly = e^{3x}\sin(7x) + 3x^2$  with undetermined coefficients.

**Solution.**

- (a) Since  $L$  is real, if  $3 + 7i$  is a repeated characteristic root of  $L$ , then  $3 - 7i$  must be a repeated characteristic root of  $L$  as well. Hence, the 4 characteristic roots must be  $3 \pm 7i, 3 \pm 7i$ .

The corresponding general solution is  $(C_1 + C_2x)e^{3x}\cos(7x) + (C_3 + C_4x)e^{3x}\sin(7x)$ .

- (b) The characteristic roots of the homogeneous DE are  $3 \pm 7i, 3 \pm 7i$  while the roots for the inhomogeneous part are  $3, 3, 3$ .

Hence, there must a particular solution of the form  $y_p = (C_1 + C_2x + C_3x^2)e^{3x}$ .

The unique values of  $C_1, C_2, C_3$  for which this is a solution of the DE need to be determined by plugging into the DE.

- (c) The characteristic roots of the homogeneous DE are  $3 \pm 7i, 3 \pm 7i$  while the roots for the inhomogeneous part are  $3 \pm 7i, 0, 0, 0$ .

Hence, there must a particular solution of the form  $C_1x^2e^{3x}\cos(7x) + C_2x^2e^{3x}\sin(7x) + C_3 + C_4x + C_5x^2$ .

The unique values of  $C_1, C_2, C_3, C_4, C_5$  for which this is a solution of the DE need to be determined by plugging into the DE.

**Problem 2.** Consider a homogeneous linear differential equation with constant real coefficients which has order 8.

- (a) Suppose  $y(x) = 7x - 2x^2e^{3x}\sin(5x)$  is a solution. Write down the general solution.
- (b) Suppose  $y(x) = 2xe^{3x} + x\cos(5x) - 5\sin(x)$  is a solution. Write down the general solution.

**Solution.**

- (a) The characteristic roots must include  $0, 0, 3 \pm 5i, 3 \pm 5i, 3 \pm 5i$ . Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is  $C_1 + C_2x + (C_3 + C_4x + C_5x^2)e^{3x}\cos(5x) + (C_6 + C_7x + C_8x^2)e^{3x}\sin(5x)$ .

- (b) The characteristic roots must include  $3, 3, \pm 5i, \pm 5i, \pm i$ . Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is  $(C_1 + C_2x)e^{3x} + (C_3 + C_4x)\cos(5x) + (C_5 + C_6x)\sin(5x) + C_7\cos(x) + C_8\sin(x)$ .

**Problem 3.**

- (a) Determine the general solution of the system 
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned}$$
- (b) Solve the IVP 
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned} \quad \text{with} \quad \begin{aligned} y_1(0) &= 4 \\ y_2(0) &= 1 \end{aligned}$$
- (c) Determine a particular solution to 
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$$
- (d) Determine the general solution to 
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$$

**Solution.**

- (a) Using  $y_2 = \frac{1}{6}(y_1 - y_1')$  (from the first equation) in the second equation, we get  $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1')$ . Simplified (and both sides multiplied by  $-6$ ), this is  $y_1'' + 3y_1' + 2y_1 = 0$ . This is a homogeneous linear DE with constant coefficients. The characteristic roots are  $-1, -2$ . Hence,  $y_1 = C_1e^{-x} + C_2e^{-2x}$ . We can then determine  $y_2$  as  $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(C_1e^{-x} + C_2e^{-2x} - (-C_1e^{-x} - 2C_2e^{-2x})) = \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$ .
- (b) From the previous part, we know  $y_1 = C_1e^{-x} + C_2e^{-2x}$  and  $y_2 = \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$ . We solve for  $C_1$  and  $C_2$  using the initial conditions:  $y_1(0) = C_1 + C_2 = 4$  and  $y_2(0) = \frac{1}{3}C_1 + \frac{1}{2}C_2 = 1$ . Solving these two equations, we find  $C_1 = 6$  and  $C_2 = -2$ . Thus, the unique solution to the IVP is  $y_1 = 6e^{-x} - 2e^{-2x}$  and  $y_2 = 2e^{-x} - e^{-2x}$ .
- (c) We proceed as in the first part to write  $y_2 = \frac{1}{6}(y_1 - y_1')$ . Using this in the second equation and simplifying, we get  $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$ . This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots of the homogeneous DE are  $-1, -2$ , while the root for the inhomogeneous part is  $3$ , there must a particular solution of the form  $y_1 = Ce^{3x}$  with undetermined coefficient  $C$ . To determine  $C$ , we plug this  $y_1$  into the DE:  $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \stackrel{!}{=} 12e^{3x}$ . Hence,  $C = \frac{3}{5}$ . Having found  $y_1 = \frac{3}{5}e^{3x}$ , we can then determine  $y_2$  as  $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(\frac{3}{5}e^{3x} - \frac{9}{5}e^{3x}) = -\frac{1}{5}e^{3x}$ .
- (d) We get the general solution by adding the particular solution (previous part) and the general solution to the corresponding homogeneous equation (first part): Hence, the general solution is  $y_1 = \frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x}$  and  $y_2 = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$ .
- Alternatively.** Here is a solution that proceeds from scratch (rather than referring to previous parts): Using  $y_2 = \frac{1}{6}(y_1 - y_1')$  (from the first equation) in the second equation, we get  $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1') - 2e^{3x}$ . Simplified (and both sides multiplied by  $-6$ ), this is  $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$ . This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots of the homogeneous DE are  $-1, -2$ , while the root for the inhomogeneous part is  $3$ , there must a particular solution of the form  $y_1 = Ce^{3x}$  with undetermined coefficient  $C$ . To determine  $C$ , we plug this  $y_1$  into the DE:  $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \stackrel{!}{=} 12e^{3x}$ . Hence,  $C = \frac{3}{5}$  and the particular solution is  $y_1 = \frac{3}{5}e^{3x}$ . The corresponding general solution is  $y_1 = \frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x}$ . We can then determine  $y_2$  as follows:

$$y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}\left(\frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x} - \left(\frac{9}{5}e^{3x} - C_1e^{-x} - 2C_2e^{-2x}\right)\right) = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}.$$

**Problem 4.**

- (a) Write the (third-order) differential equation  $y''' + 2y'' - 4y' + 5y = 2\sin(x)$  as a system of (first-order) differential equations.
- (b) Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned} \quad y_1(0) = 1, \quad y_1'(0) = 4, \quad y_2(0) = 0, \quad y_2'(0) = -1$$

Write it as a first-order initial value problem in the form  $\mathbf{y}' = M\mathbf{y} + \mathbf{f}$ ,  $\mathbf{y}(0) = \mathbf{c}$ .

**Solution.**

- (a) Write  $y_1 = y$ ,  $y_2 = y'$  and  $y_3 = y''$ .

Then, the DE translates into the first-order system 
$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -5y_1 + 4y_2 - 2y_3 + 2\sin(x) \end{cases}.$$

In matrix form, with  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ , this is  $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 4 & -2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ 2\sin(x) \end{bmatrix}$ .

- (b) Introduce  $y_3 = y_1'$  and  $y_4 = y_2'$ . Then, with  $\mathbf{y} = (y_1, y_2, y_3, y_4)$ , the given system translates into

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 7 & -3 & 0 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ e^{2x} \\ -3e^x \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}.$$

**Problem 5.** The mixtures in three tanks  $T_1, T_2, T_3$  are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from  $T_1$  is pumped into  $T_2$  at a rate of 10 gal/min and from  $T_2$  into  $T_3$  at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by  $y_i(t)$  the amount (in pounds) of salt in tank  $T_i$  at time  $t$  (in minutes). Derive a system of linear differential equations for the  $y_i$ , including initial conditions.

**Solution.** Note that at time  $t$ ,  $T_1$  contains  $10 + 15t - 10t = 10 + 5t$  gal of solution. On the other hand,  $T_2$  contains a constant amount of 10 gal, and  $T_3$   $10 + 10t$  gal of solution.

In the time interval  $[t, t + \Delta t]$ , we have:

$$\begin{aligned} \Delta y_1 &\approx 15 \cdot 2 \cdot \Delta t - 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t &\implies y_1' &= 30 - \frac{2y_1}{2 + t} \\ \Delta y_2 &\approx 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t - 10 \cdot \frac{y_2}{10} \cdot \Delta t &\implies y_2' &= \frac{2y_1}{2 + t} - y_2 \\ \Delta y_3 &\approx 10 \cdot \frac{y_2}{10} \cdot \Delta t &\implies y_3' &= y_2 \end{aligned}$$

We also have the initial conditions  $y_1(0) = 0$ ,  $y_2(0) = 0$ ,  $y_3(0) = 0$ . In matrix form, writing  $\mathbf{y} = (y_1, y_2, y_3)$ , this is

$$\mathbf{y}' = \begin{bmatrix} -\frac{2}{2+t} & 0 & 0 \\ \frac{2}{2+t} & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system of linear inhomogeneous differential equations with non-constant coefficients.

**Comment.** Because of its particularly simple structure, we actually have the techniques to solve this system. Namely, note that the first equation only involves  $y_1$ . It is a linear first-order equation which we could therefore solve using an integrating factor. With  $y_1$  determined, the second differential equation only involves  $y_2$  and is, again, a linear first-order equation. Solving it for  $y_2$ , we then get  $y_3$  by a final integration.

**Problem 6.**

- (a) What is the period and the amplitude of  $3\cos(7t) - 5\sin(7t)$ ?
- (b) Assume that the angle  $\theta(t)$  of a swinging pendulum is described by  $\theta'' + 4\theta = 0$ . Suppose  $\theta(0) = \frac{3}{10}$  and  $\theta'(0) = -\frac{4}{5}$ . What is the period and the amplitude of the resulting oscillations?
- (c) The position  $y(t)$  of a certain mass on a spring is described by  $y'' + dy' + 5y = 0$ . For which value of  $d > 0$  is the system underdamped? Critically damped? Overdamped?
- (d) A forced mechanical oscillator is described by  $y'' + 2y' + y = 25 \cos(2t)$ . As  $t \rightarrow \infty$ , what is the period and the amplitude of the resulting oscillations?
- (e) The motion of a certain mass on a spring is described by  $y'' + y' + \frac{1}{2}y = 5 \sin(t)$  with  $y(0) = 2$  and  $y'(0) = 0$ . Determine  $y(t)$ . As  $t \rightarrow \infty$ , what are the period and amplitude of the oscillations?

**Solution.**

- (a) The period is  $2\pi/7$  and the amplitude is  $\sqrt{3^2 + (-5)^2} = \sqrt{34}$ .
- (b) The characteristic equation has roots  $\pm 2i$ . Hence, the general solution to the DE is  $\theta(t) = A \cos(2t) + B \sin(2t)$ . We use the initial conditions to determine  $A$  and  $B$ :  $\theta(0) = A = \frac{3}{10}$ ,  $\theta'(0) = 2B = -\frac{4}{5}$ . Hence, the unique solution to the IVP is  $\theta(t) = \frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t)$ . In particular, the period is  $\pi$  and the amplitude is  $\sqrt{A^2 + B^2} = \sqrt{\frac{9}{100} + \frac{16}{100}} = \frac{1}{2}$ .

- (c) The characteristic equation has roots  $\frac{1}{2}(-d \pm \sqrt{d^2 - 20})$ . The system is underdamped if the solutions involve oscillations, which happens if and only if  $d^2 - 20$  (the discriminant) is negative.

Since  $d^2 - 20 < 0$  if  $d < \sqrt{20}$ , the system is underdamped for  $d \in (0, \sqrt{20})$ .

Correspondingly, the system is critically damped for  $d = \sqrt{20}$  and overdamped for  $d > \sqrt{20}$ .

- (d) The characteristic roots of the homogeneous DE are  $-1, -1$  while the roots for the inhomogeneous part are  $\pm 2i$ . Since they don't overlap, there must be a particular solution  $y_p$  of the form  $y_p = A \cos(2t) + B \sin(2t)$ .

We plug into the DE to find  $y_p'' + 2y_p' + y_p = (-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) \stackrel{!}{=} 25\cos(2t)$ .

Comparing coefficients, we get  $-3A + 4B = 25$  and  $-3B - 4A = 0$ . Solving these, we find  $A = -3$  and  $B = 4$ .

Hence,  $y_p(t) = -3 \cos(2t) + 4 \sin(2t)$  and the general solution is  $y(t) = -3 \cos(2t) + 4 \sin(2t) + (C_1 + C_2 x)e^{-t}$ .

As  $t \rightarrow \infty$ , we have  $e^{-t} \rightarrow 0$  so that  $y(t) \approx -3 \cos(2t) + 4 \sin(2t)$ .

In particular, the period is  $\pi$  and the amplitude is  $\sqrt{(-3)^2 + 4^2} = 5$ .

- (e) The characteristic roots of the homogeneous DE are  $\frac{-2 \pm \sqrt{4 - 8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$  while the roots for the inhomogeneous part are  $\pm i$ . Since there is no overlap, there must be a particular solution  $y_p$  of form  $y_p = A \cos(t) + B \sin(t)$ . By plugging into DE, we find  $A = -4$ ,  $B = -2$ .

Hence, the general solution is  $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(C_1 \cos(t/2) + C_2 \sin(t/2))$ .

We determine  $C_1$  and  $C_2$  using the initial conditions. From  $y(0) = -4 + C_1 \stackrel{!}{=} 2$ , we conclude  $C_1 = 6$ . We then compute  $y'(t) = 4\sin(t) - 2\cos(t) - \frac{1}{2}e^{-t/2}(C_1 \cos(t/2) + C_2 \sin(t/2)) + e^{-t/2}(-\frac{1}{2}C_1 \sin(t/2) + \frac{1}{2}C_2 \cos(t/2))$ . Hence,  $y'(0) = -2 - \frac{1}{2}C_1 + \frac{1}{2}C_2 \stackrel{!}{=} 0$ , from which we conclude that  $C_2 = 10$ .

Therefore, the unique solution to the IVP is  $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(6 \cos(t/2) + 10 \sin(t/2))$ .

For large  $t$ ,  $y(t) \approx -4\cos(t) - 2\sin(t)$  (since  $e^{-t/2} \rightarrow 0$ ). Hence, the period is  $2\pi$  and the amplitude is  $\sqrt{4^2 + 2^2} = \sqrt{20}$ .

**Problem 7.** The position  $y(t)$  of a certain mass on a spring is described by  $2y'' + dy' + 3y = F \sin(4\omega t)$ .

- (a) Assume first that there is no external force, i.e.  $F = 0$ . For which values of  $d$  is the system overdamped?  
 (b) Now,  $F \neq 0$  and the system is undamped, i.e.  $d = 0$ . For which values of  $\omega$ , if any, does resonance occur?

**Solution.**

- (a) The discriminant of the characteristic equation is  $d^2 - 24$ . Hence the system is overdamped if  $d^2 - 24 > 0$ , that is  $d > \sqrt{24} = 2\sqrt{6}$ .  
 (b) The natural frequency is  $\sqrt{\frac{3}{2}}$ . Resonance therefore occurs if  $4\omega = \sqrt{\frac{3}{2}}$  or, equivalently,  $\omega = \frac{1}{4}\sqrt{\frac{3}{2}}$ .

**Problem 8.**

- (a) Determine the general solution to  $y'' - 4y' + 4y = 3e^{2x}$ .  
 (b) Determine the general solution to the differential equation  $y''' - y = e^x + 7$ .  
 (c) Determine the general solution  $y(x)$  to the differential equation  $y^{(4)} + 6y''' + 13y'' = 2$ . Express the solution using real numbers only.  
 (d) Solve the initial value problem  $y'' + 2y' + y = 2e^{2x} + e^{-x}$ ,  $y(0) = -1$ ,  $y'(0) = 2$ .

**Solution.**

- (a) The characteristic equation for the corresponding homogeneous DE has roots 2, 2. The inhomogeneous part (on the right-hand side) solves a DE whose characteristic equation has root 2. Hence, by the method of undetermined coefficients, there must be a particular solution of the form  $y_p = Ax^2e^{2x}$ .

To determine  $A$ , we plug into the DE using  $y'_p = 2A(x + x^2)e^{2x}$  and  $y''_p = 2A(1 + 4x + 2x^2)e^{2x}$ :

$$y''_p - 4y'_p + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \stackrel{!}{=} 3e^{2x}. \text{ Hence, } A = \frac{3}{2} \text{ so that } y_p = \frac{3}{2}x^2e^{2x}.$$

Accordingly, the general solution is  $y(x) = (C_1 + C_2x + \frac{3}{2}x^2)e^{2x}$ .

- (b) Let us first solve the homogeneous equation  $y''' - y = 0$ . Its characteristic polynomial  $D^3 - 1 = (D - 1)(D^2 + D + 1)$  has roots 1 and  $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ . The inhomogeneous part solves a DE whose characteristic equation has roots 0, 1. Noting the repetition of the root 1, by the method of undetermined coefficients, there must be a particular solution of the form  $y_p = Ax e^x + B$ .

$$y'_p = A(x + 1)e^x, \quad y''_p = A(x + 2)e^x, \quad y'''_p = A(x + 3)e^x$$

Plugging into the DE, we get  $y'''_p - y_p = 3Ae^x - B \stackrel{!}{=} e^x + 7$ . Consequently,  $A = \frac{1}{3}$ ,  $B = -7$  so that  $y_p = -7 + \frac{1}{3}x e^x$ .

Hence, the general solution is  $y(x) = -7 + (C_1 + \frac{1}{3}x)e^x + C_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$ .

- (c) Since  $D^4 + 6D^3 + 13D^2 = D^2(D^2 + 6D + 13)$ , the characteristic equation for the corresponding homogeneous DE has roots 0, 0,  $-3 \pm 2i$ . The inhomogeneous part solves a DE whose characteristic equation has root 0. Hence, by the method of undetermined coefficients, there must be a particular solution of the form  $y_p = Ax^2$ .

Plugging into the DE, we get  $y_p^{(4)} + 6y_p''' + 13y_p'' = 26A \stackrel{!}{=} 2$ . Thus  $A = \frac{1}{13}$  so that  $y_p = \frac{1}{13}x^2$ .

Hence, the general solution is  $y(x) = \frac{1}{13}x^2 + C_1 + C_2x + C_3e^{-3x}\cos(2x) + C_4e^{-3x}\sin(2x)$ .

- (d) The characteristic equation for the associated homogeneous DE has roots  $-1, -1$ . The inhomogeneous part solves a DE whose characteristic equation has roots  $-1, 2$ .

Hence, by the method of undetermined coefficients, there must be a particular solution of the form  $y_p = Ae^{2x} + Bx^2e^{-x}$ . To find  $A, B$  we plug into the DE. [...] We find  $A = \frac{2}{9}$  and  $B = \frac{1}{2}$ .

Particular solution:  $y_p = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x}$

General solution:  $y = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} + C_1e^{-x} + C_2xe^{-x}$

Now, we use the initial values [...], to find  $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}xe^{-x}$ .

**Problem 9.**

- (a) Consider the differential equation  $x^2y'' - 4xy' + 6y = 0$ . Find all solutions of the form  $y(x) = x^r$ .  
 (b) Determine the general solution of  $x^2y'' - 4xy' + 6y = x^3$ .

**Solution.**

- (a) Plugging  $y(x) = x^r$  into the DE, we get  $x^2r(r - 1)x^{r-2} - 4rxr^{r-1} + 6x^r = [r(r - 1) - 4r + 6]x^r \stackrel{!}{=} 0$ .

Since  $r(r - 1) - 4r + 6 = (r - 2)(r - 3)$ , we find the solutions  $x^2$  and  $x^3$ . Since this is a second-order equation and our solutions are independent, there can be no further solutions.

- (b) We can find a particular solution to this inhomogeneous DE using the method of variation of parameters/constants. From the first part, we know that the corresponding homogeneous DE has the solutions  $y_1 = x^2$ ,  $y_2 = x^3$ . The Wronskian of these is  $W(x) = y_1y'_2 - y'_1y_2 = x^4$ .

Put the DE in the form  $y'' - 4x^{-1}y' + 6x^{-2}y = f(x)$  with  $f(x) = x$ . Then, by the method of variation of parameters, a particular solution is given by

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx = -x^2 \int 1 dx + x^3 \int \frac{1}{x} dx = -x^3 + x^3 \ln|x|.$$

Hence, the general solution is  $y(x) = C_1x^2 + (C_2 + \ln|x|)x^3$ .