

Midterm #2 – Practice

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $3 + 7i$ is a repeated characteristic root of L .

- What is the general solution to $Ly = 0$?
- Write down the simplest form of a particular solution y_p of the DE $Ly = 7x^2e^{3x}$ with undetermined coefficients.
- Write down the simplest form of a particular solution y_p of the DE $Ly = e^{3x}\sin(7x) + 3x^2$ with undetermined coefficients.

Problem 2. Consider a homogeneous linear differential equation with constant real coefficients which has order 8.

- Suppose $y(x) = 7x - 2x^2e^{3x}\sin(5x)$ is a solution. Write down the general solution.
- Suppose $y(x) = 2xe^{3x} + x\cos(5x) - 5\sin(x)$ is a solution. Write down the general solution.

Problem 3.

- Determine the general solution of the system
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned}$$
- Solve the IVP
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned} \quad \text{with} \quad \begin{aligned} y_1(0) &= 4 \\ y_2(0) &= 1 \end{aligned}$$
- Determine a particular solution to
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$$
- Determine the general solution to
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$$

Problem 4.

- Write the (third-order) differential equation $y''' + 2y'' - 4y' + 5y = 2\sin(x)$ as a system of (first-order) differential equations.
- Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} & y_1(0) &= 1, \quad y_1'(0) = 4, \quad y_2(0) = 0, \quad y_2'(0) = -1 \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned}$$

Write it as a first-order initial value problem in the form $\mathbf{y}' = M\mathbf{y} + \mathbf{f}$, $\mathbf{y}(0) = \mathbf{c}$.

Problem 5. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 10 gal/min and from T_2 into T_3 at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i , including initial conditions.

Problem 6.

- (a) What is the period and the amplitude of $3\cos(7t) - 5\sin(7t)$?
- (b) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the period and the amplitude of the resulting oscillations?
- (c) The position $y(t)$ of a certain mass on a spring is described by $y'' + dy' + 5y = 0$. For which value of $d > 0$ is the system underdamped? Critically damped? Overdamped?
- (d) A forced mechanical oscillator is described by $y'' + 2y' + y = 25 \cos(2t)$. As $t \rightarrow \infty$, what is the period and the amplitude of the resulting oscillations?
- (e) The motion of a certain mass on a spring is described by $y'' + y' + \frac{1}{2}y = 5 \sin(t)$ with $y(0) = 2$ and $y'(0) = 0$. Determine $y(t)$. As $t \rightarrow \infty$, what are the period and amplitude of the oscillations?

Problem 7. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F \sin(4\omega t)$.

- (a) Assume first that there is no external force, i.e. $F = 0$. For which values of d is the system overdamped?
- (b) Now, $F \neq 0$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?

Problem 8.

- (a) Determine the general solution to $y'' - 4y' + 4y = 3e^{2x}$.
- (b) Determine the general solution to the differential equation $y''' - y = e^x + 7$.
- (c) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 6y''' + 13y'' = 2$. Express the solution using real numbers only.
- (d) Solve the initial value problem $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

Problem 9.

- (a) Consider the differential equation $x^2y'' - 4xy' + 6y = 0$. Find all solutions of the form $y(x) = x^r$.
- (b) Determine the general solution of $x^2y'' - 4xy' + 6y = x^3$.