Midterm #2

MATH 238 — Differential Equations I
Wednesday, Nov 13, 2024

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 32 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (10 points) Determine the general solution of the following system:
$$\begin{aligned} y_1' &= 3y_1 + y_2 \\ y_2' &= y_1 + 3y_2 - 2e^x \end{aligned}$$

Solution. Using $y_2 = y_1' - 3y_1$ (from the first equation) in the second equation, we get $y_1'' - 3y_1' = y_1 + 3(y_1' - 3y_1) + 6e^x$.

Simplified, this is $y_1'' - 6y_1' + 8y_1 = 6e^x$. This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots for the homogeneous DE are $2, 4$, while the root for the inhomogeneous part is 1, there must a particular solution of the form $y_1 = Ae^x$. For this y_1, $y_1'' - 6y_1' + 8y_1 = (1 - 6 + 8)Ae^x = 3Ae^x \overset{!}{=} -2e^x$. Hence, $A = -\frac{2}{3}$ and the particular solution is $y_1 = -\frac{2}{3}e^x$. The corresponding general solution is $y_1 = -\frac{2}{3}e^x + C_1e^{2x} + C_2e^{4x}$.

Correspondingly, $y_2 = y_1' - 3y_1 = -\frac{2}{3}e^x + 2C_1e^{2x} + 4C_2e^{4x} - 3\left(-\frac{2}{3}e^x + C_1e^{2x} + C_2e^{4x}\right) = \frac{4}{3}e^x - C_1e^{2x} + C_2e^{4x}$.

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Problem 2. (6 points) The mixtures in two tanks T_1, T_2 are kept uniform by stirring. Brine containing 3 lb of salt per gallon enters the first tank at a rate of 10 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 8 gal/min, and also from T_2 into T_1 at a rate of 5 gal/min. Initially, tank T_1 is filled with 25 gal water containing 4 lb salt, and tank T_2 with 20 gal pure water.

Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions. (Do *not* attempt to solve the system.)

Solution. Note that after t minutes, T_1 contains $25+10t-8t+5t=25+7t$ gal of solution while T_2 contains $20+8t+-5t=20+3t$ gal of solution. In the time interval $[t, t+\Delta t]$, we have:

$$\Delta y_1 \approx 10\cdot 3\cdot \Delta t - 8\cdot\frac{y_1}{25+7t}\cdot\Delta t + 5\cdot\frac{y_2}{20+3t}\cdot\Delta t \quad\Longrightarrow\quad y_1' = 30 - \frac{8}{25+7t}y_1 + \frac{5}{20+3t}y_2$$

$$\Delta y_2 \approx 8\cdot\frac{y_1}{25+7t}\cdot\Delta t - 5\cdot\frac{y_2}{20+3t}\cdot\Delta t \quad\Longrightarrow\quad y_2' = \frac{8}{25+7t}y_1 - \frac{5}{20+3t}y_2$$

The initial conditions are $y_1(0)=4,\ y_2(0)=0$.

Optional: in matrix form, writing $\boldsymbol{y}=(y_1, y_2)$, this takes the form

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{8}{25+7t} & \frac{5}{20+3t} \\ \frac{8}{25+7t} & -\frac{5}{20+3t} \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 30 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Problem 3. (4 points) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta''+25\theta=0$. Suppose $\theta(0)=3,\ \theta'(0)=-5$. What are the period and the amplitude of the resulting oscillations?

Solution. The characteristic equation has roots $\pm 5i$. Hence, the general solution to the DE is $\theta(t) = A\cos(5t) + B\sin(5t)$.

We use the initial conditions to determine A and B: $\theta(0) = A \overset{!}{=} 3$. $\theta'(0) = 5B \overset{!}{=} -5$.

Hence, the unique solution to the IVP is $\theta(t) = 3\cos(5t) - \sin(5t)$.

In particular, the period is $2\pi/5$ and the amplitude is $\sqrt{A^2+B^2} = \sqrt{3^2+(-1)^2} = \sqrt{10}$.

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Problem 4. (4 points) Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $2-3i$ is a repeated characteristic root of L.

(a) What is the general solution to $Ly=0$?

(b) Write down the simplest form of a particular solution y_p of the DE $Ly=2xe^x-5e^{2x}\sin(3x)$ with undetermined coefficients.

Solution. Since L is real, if $2-3i$ is a repeated characteristic root of L, then $2+3i$ must be a repeated characteristic root of L as well. Hence, the 4 characteric roots must be $2\pm3i, 2\pm3i$.

(a) The general solution is $(C_1+C_2x)e^{2x}\cos(3x)+(C_3+C_4x)e^{2x}\sin(3x)$.

(b) The roots for the inhomogeneous part are $1, 1, 2\pm3i$.

Hence, there must a particular solution of the form $(C_1+C_2x)e^x+C_3x^2e^{2x}\cos(3x)+C_4x^2e^{2x}\sin(3x)$.

Problem 5. (4 points) The position $y(t)$ of a certain mass on a spring is described by $3y''+dy'+5y=F\cos(\omega t)$.

(a) Assume first that there is no external force, i.e. $F=0$. For which values of d is the system underdamped?

(b) Now, $F\neq0$ and the system is undamped, i.e. $d=0$. For which values of ω, if any, does resonance occur?

Solution.

(a) The discriminant of the characteristic equation is d^2-60. Hence the system is underdamped if $d^2-60<0$, that is $d<\sqrt{60}$.

(b) The natural frequency is $\sqrt{\frac{5}{3}}$. Resonance therefore occurs if $\omega=\sqrt{\frac{5}{3}}$.

Problem 6. (4 points) Consider the following system of initial value problems:

$$\begin{array}{l} y_1''+3y_1=2y_2'+4 \\ y_2''+5y_2=6y_1' \end{array} \qquad y_1(0)=7,\ y_1'(0)=0,\ y_2(0)=6,\ y_2'(0)=-1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}'=M\boldsymbol{y}+\boldsymbol{f},\ \boldsymbol{y}(0)=\boldsymbol{c}$.

Solution. Introduce $y_3=y_1'$ and $y_4=y_2'$. Then, the given system translates into

$$\boldsymbol{y}'=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 0 & 2 \\ 0 & -5 & 6 & 0 \end{bmatrix}\boldsymbol{y}+\begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \qquad \boldsymbol{y}(0)=\begin{bmatrix} 7 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

(extra scratch paper)

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