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UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

DEPARTMENT OF MATHEMATICS

MATH 286 SECTION X1 – Introduction to Differential Equations Plus

FINAL EXAMINATION

DECEMBER 17, 2013

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INSTRUCTIONS

- This exam is three (3) hours long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- EXPLAIN YOUR WORK! Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- GOOD LUCK!

PLEASE NOTE: "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."

Question:	1	2	3	4	5	6	7	Total
Points:	14	25	9	9	14	15	14	100
Score:								

SOME USEFUL FORMULAS:

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \dots$$

$$\mathbf{x}(t) = \Phi(t)\Phi(a)^{-1}\mathbf{x}(a) + \Phi(t)\int_a^t \Phi(s)^{-1}\mathbf{f}(s)ds$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

1. Let $f(t)$ be the π-periodic function defined by

$$f(t) = \begin{cases} 1 & -\frac{\pi}{2} < t < 0, \\ t & 0 \le t \le \frac{\pi}{2}. \end{cases}$$

(a) (2 points) Sketch the graph of f over a few periods.

(b) (6 points) Let $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L})$ be the Fourier series for f. Calculate the Fourier coefficients a_0, a_n and b_n $(n \ge 1)$.

Solution:

$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(t)dt = \frac{2}{\pi}\Big(\frac{\pi}{2} + \frac{\pi^2}{8}\Big) = 1 + \frac{\pi}{4},$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\cos(2nt)dt = \frac{2}{\pi}\int_{-\pi/2}^{0}\cos(2nt)dt + \frac{2}{\pi}\int_{0}^{\pi/2} t\cos(2nt)dt$$

$$= \frac{\sin(2nt)}{n\pi}\Big|_{-\pi/2}^{0} + \frac{2}{\pi}\Big(\frac{t\sin(2nt)}{2n} + \frac{\cos(2nt)}{4n^2}\Big)\Big|_{0}^{\pi/2}$$

$$= \frac{\cos(n\pi) - 1}{2\pi n^2},$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\sin(2nt)dt = \frac{2}{\pi}\int_{-\pi/2}^{0}\sin(2nt)dt + \frac{2}{\pi}\int_{0}^{\pi/2} t\sin(2nt)dt$$

$$= \frac{-\cos(2nt)}{n\pi}\Big|_{-\pi/2}^{0} + \frac{2}{\pi}\Big(\frac{-t\cos(2nt)}{2n} + \frac{\sin(2nt)}{4n^2}\Big)\Big|_{0}^{\pi/2}$$

$$= \frac{\cos(n\pi) - 1}{n\pi} - \frac{\cos(n\pi)}{2n}.$$

(c) (2 points) Does the Fourier series for f converge to $f(t)$ at every point t? What does the Fourier series converge to when $t = 0$?

> **Solution:** Since f is piecewise smooth and has jump discontinuities, the FS for f does not converge at every point t. For example, at $t = 0$ the FS converges to $\frac{f(0^+)+f(0^-)}{2} = 1/2 \neq f(0)$.

(d) (4 points) A 1 kg cart is connected to a wall by a spring with unknown spring constant $k > 0$ N/m, and is periodically forced by $f(t)$ Newtons (where f is the periodic function defined above). Assuming there is no friction in the system, the resulting equation of motion for the displacement $x(t)$ of the cart from rest is given by

$$x'' + kx = f(t).$$

Find all values of k that will cause resonance in the forced mechanical system.

> **Solution:** When the Fourier Series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(2nt) + b_n \sin(2nt)$ for f contains a non-zero sine or cosine component with frequency equal to the natural frequency $\omega_0 = \sqrt{k}$ of the system, there will be resonance. Since all the b_n's (and all the odd a_n's) are non-zero, this happens precisely when
>
> $$\sqrt{k} = 2n \iff k = 4n^2 \qquad (n = 1, 2, 3, \ldots).$$

2. In this multi-part problem, we will derive the solution to a one-dimensional heat equation with *mixed boundary conditions* (with one endpoint held at a fixed temperature, and the other endpoint insulated). For the remainder of this problem, let $L > 0$ be fixed.

   (a) (6 points) Consider the constant function

   $$f(x) = 100 \quad \text{defined on the interval } [0, 2L].$$

   Sketch the graph of the $4L$-**periodic** *odd extension* of f and compute its Fourier sine series.

   Solution: The expansion we seek is $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2L}$, where

   $$b_n = \frac{2}{2L}\int_0^{2L} f(x)\sin\frac{n\pi x}{2L}dx = \frac{100}{L}\int_0^{2L}\sin\frac{n\pi x}{2L}dx$$
   $$= \frac{-200}{n\pi}\cos\frac{n\pi x}{2L}\Big|_0^{2L} = \begin{cases} 0 & n \text{ is even} \\ \frac{400}{n\pi} & n \text{ is odd} \end{cases}$$

   (b) (5 points) Consider the following eigenvalue problem for the function $X(x)$ on the interval $[0, L]$:

   $$X'' + \lambda X = 0; \quad X(0) = X'(L) = 0.$$

   Show that the λ is an eigenvalue if and only if $\lambda = \lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$, where $n = 1, 2, 3, \ldots$. For each λ_n, write down the corresponding eigenfunction $X_n(x)$. (**Note:** You may assume without proof that all the eigenvalues are *positive*.)

   Solution: Since $\lambda > 0$, the general solution to this ODE is

   $$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x).$$

   From the BC's, we get

   $$c_1 = 0, \quad \& \quad \sqrt{\lambda}c_2\cos(\sqrt{\lambda}2L) = 0 \implies \sqrt{\lambda}2L = (2n-1)\pi \iff \lambda = \frac{(2n-1)^2\pi^2}{4L^2},$$

   where $n = 1, 2, \ldots$. The corresponding eigenfunction is $X_n(x) = \sin\frac{(2n-1)\pi x}{2L}$.

(c) (3 points) A laterally insulated metal rod of length L (with thermal diffusivity $k = 2$) is heated to a uniform temperature of 100 degrees Celsius. At time $t = 0$, the left end of the rod ($x = 0$) is placed in an ice bath at 0 degrees Celsius, and the right end ($x = L$) is insulated so that no heat flows in or out at this end. If $u(x, t)$ denotes the temperature (in degrees Celsius) of the rod at position $0 < x < L$ and time $t > 0$, then u satisfies the one-dimensional heat equation

$$u_t = 2u_{xx} \qquad (0 < x < L,\ t > 0).$$

Write down the Boundary Conditions and Initial Condition for this problem.

Solution:

$$u(0, t) = 0 = u_x(L, t) = 0 \quad \& \quad u(x, 0) = 100.$$

(d) (4 points) Using the method of **separation of variables**, show that if

$$u(x, t) = X(x)T(t)$$

is a solution to the above heat equation satisfying the boundary conditions from part (c), then $X(x)$ must be a solution to the eigenvalue problem in part (b).

Solution: If $u(x, t) = X(x)T(t)$, then

$$u_t = 2u_{xx} \iff XT' = 2X''T \iff \frac{X''}{X} = \frac{T'}{2T} = -\lambda \qquad (\lambda \in \mathbb{R}).$$

This together with the BCs above yields the two ODEs

$$X'' + \lambda X = 0; \qquad X(0) = X'(L) = 0$$

and

$$T' + 2\lambda T = 0.$$

(e) (3 points) For each eigenfunction $X_n(x)$ from part (b), find the corresponding solution $T_n(t)$.

Solution:

$$T_n' + 2\frac{(2n-1)^2\pi^2}{4L^2}T_n = 0 \implies T_n(t) = \exp\Big(-2\frac{(2n-1)^2\pi^2 t}{4L^2}\Big)$$

(f) (4 points) Let

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n X_n(x) T_n(t).$$

Then $u(x,t)$ satisfies the heat equation and boundary conditions from part (c). Find the constants α_n so that $u(x,t)$ also satisfies the initial condition $u(x,0)$. (**Hint:** Use part (a).)

Solution: We have

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L} \exp\Big(-2\frac{(2n-1)^2\pi^2 t}{4L^2}\Big),$$

so

$$100 = u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L}.$$

This tells us that

$$\alpha_n = b_{2n-1} \qquad (n = 1, 2, \ldots).$$

3. (9 points) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x+3y}{x-y}; \qquad y(1) = 0.$$

An implicit equation for $y(x)$ is fine.

Solution: Let $v = \frac{y}{x}$. Then $\frac{dy}{dx} = xv' + v$ and the equation becomes

$$xv' + v = \frac{1+3v}{1-v} \iff xv' = \frac{1+2v+v^2}{1-v} \iff \frac{(1-v)dv}{(v+1)^2} = \frac{dx}{x}.$$

Integrating, this gives

$$\frac{-2}{v+1} - \ln|v+1| = \ln|x| + C \iff \frac{-2}{y/x+1} - \ln|y/x+1| = \ln|x| + C.$$

Since $y(1) = 0$, we get $-2 - 0 = 0 + C$, so the solution is

$$\frac{-2}{y/x+1} - \ln|y/x+1| = \ln|x| - 2.$$

4. Consider the ODE

$$y + (2x - e^y)\frac{dy}{dx} = 0.$$

(a) (3 points) Is this equation exact?

Solution: Let $M = y$ and $N = 2x - e^y$. Clearly $M_y = 1 \neq 2 = N_x$, so the equation is not exact.

(b) (6 points) Find an implicit expression for the general solution to this ODE. (**HINT:** Multiply the above ODE by y and then check for exactness).

Solution: Multiplying by y, the ODE becomes

$$y^2 + (2xy - ye^y)\frac{dy}{dx} = 0.$$

Let $M = y^2$ and $N = 2xy - ye^y$. Then $M_y = 2y = N_x$, so the equation is exact. To solve the ODE, we find an $F(x, y)$ such that $F_x = M$ and $F_y = N$. Then $F(x, y) = C$ will be an implicit solution. Now,

$$F(x, y) = \int M dx = \int y^2 dx = xy^2 + g(y).$$

To find $g(y)$, note that

$$2xy + g'(y) = F_y = N = 2xy - ye^y \implies g'(y) = -ye^y \implies g(y) = -ye^y + e^y.$$

Thus, $xy^2 + -ye^y + e^y = C$ is a solution.

5. (a) (3 points) Find the general solution to the ODE

$$y'' - 10y' + 21y = 0.$$

Solution: Let $P(r) = r^2 - 10r + 21 = (r-7)(r-3)$ be the characteristic polynomial. Then the general solution is

$$y(x) = c_1 e^{7x} + c_2 e^{3x}.$$

(b) (6 points) Solve the initial value problem

$$y'' - 10y' + 21y = e^{3x} + e^{x}; \qquad y(0) = y'(0) = 0.$$

Solution: Let $y_p(x) = Axe^{3x} + Be^{x}$ be a trial particular solution. Plugging this in, we get

$$6Ae^{3x} + 9Axe^{3x} + Be^{x} - 10(Ae^{3x} + 3Axe^{3x} + Be^{x}) + 21(Axe^{3x} + Be^{x}) = e^{3x} + e^{x}.$$

This gives

$$6A - 10A = 1 \implies A = -\frac{1}{4} \quad \& \quad B - 10B + 21B = 1 \implies B = \frac{1}{12}.$$

The general solution to this ODE is then

$$y(x) = c_1 e^{7x} + c_2 e^{3x} - \frac{1}{4}xe^{3x} + \frac{1}{12}e^{x}.$$

Plugging in the IC's we get $c_1 = \frac{5}{48}$, $c_2 = \frac{-9}{48}$.

$$y(x) = \frac{5}{48}e^{7x} - \frac{9}{48}e^{3x} - \frac{1}{4}xe^{3x} + \frac{1}{12}e^{x}$$

(c) (5 points) Find the general solution to the ODE

$$y^{(5)} + 8y^{(3)} + 16y' = 1 + (1 + e^{x})\cos(2x).$$

(**NOTE:** For part (c), you do not need to evaluate the undetermined coefficients $A, B, \ldots$).

Solution: Here, $P(r) = r^5 + 8r^3 + 16r = r(r^2+4)^2 = r(r+2i)^2(r-2i)^2$. Therefore we have

$$y_c(x) = c_1 + c_2\cos(2x) + c_3\sin(2x) + c_4 x\cos(2x) + c_5 x\sin(2x),$$

$$y_p(x) = Ax + x^2(B\cos(2x) + C\sin(2x)) + De^x\cos(2x) + Fe^x\sin 2x,$$

and the general solution is of the form $y = y_c + y_p$.

6. Consider the following second order ODE for the function $y(t)$.

$$t^2y'' + ty' + y = 0 \qquad (t > 0).$$

(a) (3 points) Verify that $y_1(t) = \cos(\ln t)$ and $y_2(t) = \sin(\ln t)$ are two linearly independent solutions to this ODE.

Solution:

$$t^2y_1'' + ty_1' + y_1 = t^2(t^{-2}\sin(\ln t) + -t^{-2}\cos\ln t) + t(-t^{-1}\sin(\ln t)) + \cos(\ln t) = 0.$$

$$t^2y_2'' + ty_2' + y_2 = t^2(-t^{-2}\cos(\ln t) + -t^{-2}\sin\ln t) + t(t^{-1}\cos(\ln t)) + \sin(\ln t) = 0.$$

These functions are obviously linearly independent.

(b) (3 points) Using the substitutions $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, rewrite this ODE as an equivalent two-dimensional first order system of the form

$$\mathbf{x}' = P(t)\mathbf{x} \quad \text{where} \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \& \quad P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}.$$

Solution:

$$x_1' = y' = x_2 \quad \& \quad x_2' = y'' = \frac{-y - ty'}{t^2} = -t^{-2}x_1 - t^{-1}x_2.$$

$$\implies \mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -t^{-2} & -t^{-1} \end{bmatrix} \mathbf{x}(t)$$

(c) (3 points) Write down a fundamental matrix $\Phi(t)$ for the system in part (b). (**Hint:** Use part (a).)

Solution:

$$\Phi(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1}\sin(\ln t) & t^{-1}\cos(\ln t) \end{bmatrix}.$$

(d) (6 points) Solve the *non-homogeneous* the initial value problem

$$\mathbf{x}' = P(t)\mathbf{x} + \begin{bmatrix} t^{-1} \\ 0 \end{bmatrix}; \qquad \mathbf{x}(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution: We use the variation of parameters formula, noting that

$$\Phi(s)^{-1} = \begin{bmatrix} \cos(\ln s) & -s\sin(\ln s) \\ \sin(\ln s) & s\cos(\ln s) \end{bmatrix} \quad \& \quad \Phi(1)^{-1} = I.$$

$$\begin{aligned}
\mathbf{x}(t) &= \Phi(t)\Phi(1)^{-1}\mathbf{x}(1) + \Phi(t)\int_1^t \Phi(s)^{-1}\mathbf{f}(s)ds \\
&= \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1}\sin(\ln t) & t^{-1}\cos(\ln t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1}\sin(\ln t) & t^{-1}\cos(\ln t) \end{bmatrix} \int_1^t \begin{bmatrix} \cos(\ln s) & -s\sin(\ln s) \\ \sin(\ln s) & s\cos(\ln s) \end{bmatrix} \begin{bmatrix} s^{-1} \\ 0 \end{bmatrix} ds \\
&= \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1}\sin(\ln t) & t^{-1}\cos(\ln t) \end{bmatrix} \int_1^t \begin{bmatrix} s^{-1}\cos(\ln s) \\ s^{-1}\sin(\ln s) \end{bmatrix} ds \\
&= \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1}\sin(\ln t) & t^{-1}\cos(\ln t) \end{bmatrix} \begin{bmatrix} \sin(\ln t) \\ 1 - \cos(\ln t) \end{bmatrix} \\
&= \begin{bmatrix} \sin(\ln t) \\ t^{-1}(\cos(\ln t) - 1) \end{bmatrix}
\end{aligned}$$

7. The matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

has one eigenvalue λ with multiplicity 3.

(a) (6 points) Find the eigenvalue λ and all eigenvectors associated to λ.

Solution: The characteristic equation is:

$$0 = P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{bmatrix}$$
$$= -(1+\lambda)((1-\lambda)(-3-\lambda)+4) - (1+\lambda)(1+2\lambda+\lambda^2)$$
$$= -(\lambda+1)^3.$$

Therefore $\lambda = -1$ and if $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an eigenvector, then

$$(A - I)v = 0 \iff A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}.$$

(b) (2 points) What is the defect of this eigenvalue?

Solution: Since (up to scaling) there is only one eigenvector, the defect is $3 - 1 = 2$.

(c) (6 points) Find the general solution to the 3 dimensional linear system

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \mathbf{x}(t).$$

Solution: We must build a length three chain $\{v_1, v_2, v_3\}$ of generalized eigenvectors based at an eigenvector $v_1 = [1, 0, 0]^T$ (taking this as our choice for v_1). Then we have for $v_2 = [a, b, c]^T$,

$$v_1 = (A + I)v_2 \iff \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies c = 1 \;\&\; b = 2.$$

Thus we may take $v_2 = [0, 2, 1]^T$. Next, if $v_3 = [a, b, c]^T$, then

$$v_2 = (A + I)v_3 \iff \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \implies c = 1 \;\&\; b = 3.$$

Thus, we can take $v_3 = [0, 3, 1]^T$. The general solution is then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t),$$

where

$$\mathbf{x}_1(t) = e^{-t}v_1, \quad \mathbf{x}_2(t) = e^{-t}(tv_1 + v_2), \quad \mathbf{x}_3(t) = e^{-t}(\frac{t^2}{2}v_1 + tv_2 + v_3).$$

(*Extra work space.*)