Preparing the Final

MATH 286 — Differential Equations Plus
May 9, 2014

Read Euler, read Euler, he is the master of us all.
— Pierre-Simon Laplace (1749–1827) —

Problem 1.

(a) Find the Fourier series of the function of period 2 characterized by

$$f(t) = \begin{cases} t, & \text{for } 0 \leqslant t < 1, \\ t+2, & \text{for } 1 \leqslant t < 2. \end{cases}$$

(b) Let $g(t)$ be the sum of the Fourier series you just calculated. Sketch the graph of $g(t)$. What are $g(0)$, $g(1)$ and $g(2)$? Explain the general phenomenon.

Solution.

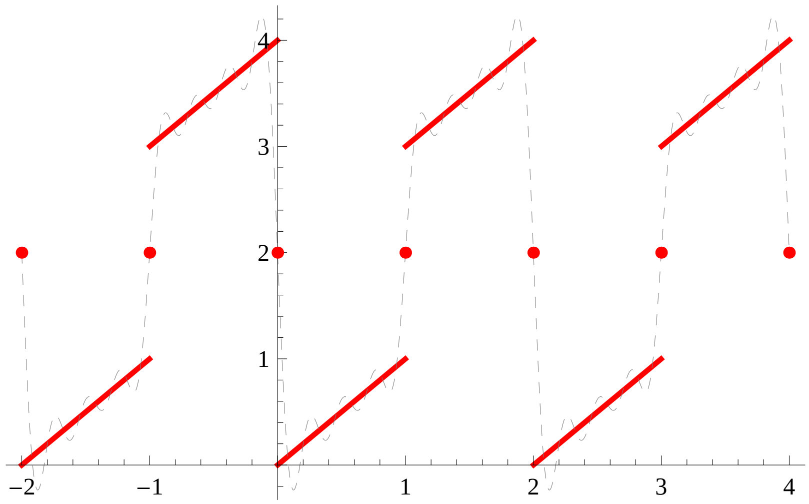
(a) We compute:

$$\begin{aligned} a_0 &= \int_0^2 f(t)\mathrm{d}t = \int_0^1 t\,\mathrm{d}t + \int_1^2 (t+2)\mathrm{d}t = 4 \\ a_m &= \int_0^2 f(t)\cos(m\pi t)\mathrm{d}t = \int_0^2 t\cos(m\pi t)\mathrm{d}t + \int_1^2 2\cos(m\pi t)\mathrm{d}t = \ldots = 0 \\ b_m &= \int_0^2 f(t)\sin(m\pi t)\mathrm{d}t = \int_0^2 t\sin(m\pi t)\mathrm{d}t + \int_1^2 2\sin(m\pi t)\mathrm{d}t = \ldots = \frac{2((-1)^m - 2)}{m\pi} \end{aligned}$$

Note that $a_m = 0$ for $m > 0$ follows (and we could have avoided the calculation by noticing) from the fact that $f(t) - \frac{a_0}{2} = f(t) - 2$ is an odd function (make a plot to convince yourself!). The Fourier series is:

$$2 + \sum_{m=1}^{\infty} \frac{2((-1)^m - 2)}{m\pi}\sin(m\pi t)$$

(b) Graph of $g(t)$ in red (with an approximation in light dashed gray):



We have $g(0) = g(1) = g(2) = 2$. These values are the averages between the limit from the left and the limit from the right (you can see from the dashed approximation of the Fourier series why this general behaviour of Fourier series makes sense). □

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Problem 2. A mass-spring system is described by the equation

$$mx'' + x = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

(a) For which m does pure resonance occur?

(b) Find the general solution when $m = 1/9$.

Solution.

(a) Characteristic equation: $mr^2 + 1 = 0$ with roots $r = \pm i\sqrt{1/m}$.

Natural frequency is $\sqrt{1/m}$.

The external frequencies are $n/3$ where n is an odd positive integer.

$\sqrt{1/m} = n/3 \Longleftrightarrow m = 9/n^2$ (since $m > 0$)

Pure resonance occurs if $m = 9/n^2$ for an odd integer $n \geqslant 1$ (that is, $m = 9, 1, 9/25, 9/49, \ldots$).

(b) In this case, the natural frequency is 3 and we have pure resonance because $3 = n/3$ for $n = 9$. For $n \neq 9$ we solve

$$\frac{1}{9} x'' + x = \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

This has a solution of the form $x_p = A\cos\left(\frac{nt}{3}\right) + B\sin\left(\frac{nt}{3}\right)$ where A, B are undetermined. Plugging into the DE:

$$\frac{1}{9} x_p'' + x_p = A\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\cos\left(\frac{nt}{3}\right) + B\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\sin\left(\frac{nt}{3}\right) \overset{!}{=} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$$

It follows that $A = 0$ (we could have seen that coming...) and

$$B = \frac{1}{n^2\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)} = \frac{81}{n^2(81 - n^2)}, \qquad x_p = \frac{81}{n^2(81 - n^2)} \sin\left(\frac{nt}{3}\right).$$

The case $n = 9$ has to be done separately: because of resonance there now exists a solution of the form

$$x_p = At\cos(3t) + Bt\sin(3t).$$

Plugging into the DE:

$$\frac{1}{9} x_p'' + x_p = \frac{2}{3} B\cos(3t) - \frac{2}{3} A\sin(3t) \overset{!}{=} \frac{1}{81}\sin(3t)$$

It follows that $B = 0$ and $A = -\frac{1}{54}$. So $x_p = -\frac{1}{54} t\cos(3t)$. By superposition it follows that

$$\frac{1}{9} x'' + x = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right) \qquad \text{has solution} \qquad x_p = -\frac{1}{54} t\cos(3t) + \sum_{\substack{n=1\\ n \text{ odd},\, n \neq 9}} \frac{81}{n^2(81 - n^2)} \sin\left(\frac{nt}{3}\right).$$

The general solution is $x(t) = x_p(t) + A\cos(3t) + B\sin(3t)$. □

Problem 3. Let $f(t) = 1$ for $t \in (0, L)$.

(a) Extend $f(t)$ to an odd $2L$-periodic function $f_o(t)$. Sketch the graph of the sum of the Fourier series of $f_o(t)$.

(b) Calculate the Fourier series of $f_o(t)$ with period $2L$. (This is also known as the Fourier sine series of $f(t)$.)

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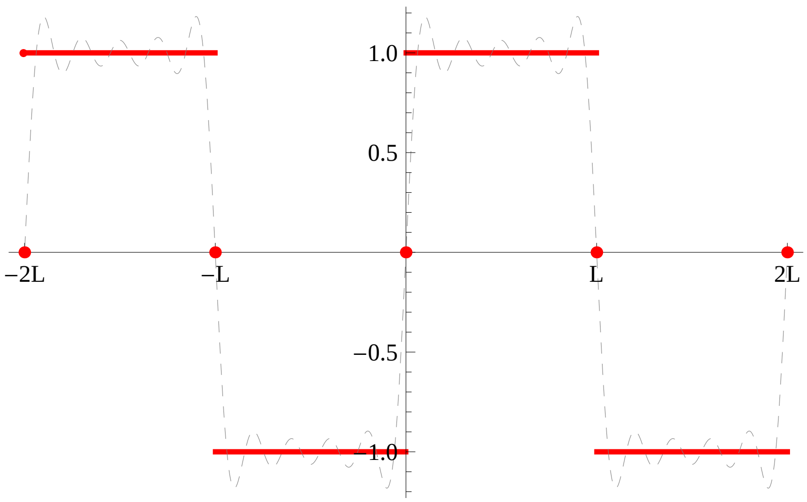
(c) Explain, using the heat equation as an example, why it can be useful to write a *constant function* as an infinite sum of sine terms.

Solution.

(a) Sketch of the Fourier series in red (with an approximation in light dashed gray):



(b) The odd $2L$-periodic extension of $f(t)$ takes the values $f(t) = \begin{cases} -1 & \text{for } t \in (-L,0) \\ +1 & \text{for } t \in (0,L) \end{cases}$.

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t = \frac{2}{L}\left[-\frac{L}{\pi n}\cos\left(\frac{n\pi t}{L}\right)\right]_0^L = \frac{2}{\pi n}[1-\cos(n\pi)] \\ &= \frac{2}{\pi n}[1-(-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Thus the Fourier series is:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin\left(\frac{n\pi t}{L}\right)$$

(c) When solving the heat equation we found that the equations

$$\begin{aligned} u_t &= k u_{xx} \qquad t>0, \quad x\in(0,L) \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

have the solutions $u_n(x,t) = e^{-\frac{n^2\pi^2}{L^2}kt}\sin\left(\frac{n\pi}{L}x\right)$. The superposition $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$ then satisfies

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right).$$

This is the initial temperature distribution and the c_n are its Fourier (sine) coefficients! In applications, $u(x,0)$ is often given to us; for instance, $u(x,0)=1$. In this case, we need to know the Fourier sine expansion of 1 on $(0,L)$ in order to find the coefficients c_n. □

Problem 4. For which values of λ does the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$$

have nonzero solutions? Find all these solutions. Make sure to consider all cases.

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Solution. To solve this EVP, we distinguish three cases:

- $\boldsymbol{\lambda < 0}$**.** Then the roots are the real numbers $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution to the EVP for $\lambda < 0$.

- $\boldsymbol{\lambda = 0}$**.** Now the general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.

- $\boldsymbol{\lambda > 0}$**.** Now the roots are $\pm i\sqrt{\lambda}$ and $y(x) = A\cos\left(\sqrt{\lambda}\,x\right) + B\sin\left(\sqrt{\lambda}\,x\right)$. $y'(0) = B\sqrt{\lambda} = 0$ implies $B = 0$. Then $y(3) = A\cos\left(3\sqrt{\lambda}\right) = 0$. Note that $\cos\left(3\sqrt{\lambda}\right) = 0$ is true if and only if $3\sqrt{\lambda} = \frac{(2n+1)\pi}{2}$ for some integer n. In that case, $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, this means that the only nonzero solutions are the functions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$, corresponding to $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ (why did we include $n = 0$ but excluded $n = -1, -2, \ldots$?!). □

Problem 5. Find the solution $u(x,t)$, for $0 < x < 3$ and $t \geqslant 0$, to the heat conduction problem

$$2u_t = u_{xx}, \quad u_x(0,t) = 0, \; u(3,t) = 0, \quad u(x,0) = 2\cos\left(\frac{\pi x}{2}\right) + 7\cos\left(\frac{3\pi x}{2}\right).$$

Derive your solution using separation of variables (at some step you may refer to the previous problem). Don't rely on a formula.

Solution. We look for solutions to (that's the homogeneous/linear parts of the problem at hand)

$$2u_t = u_{xx}, \quad u_x(0,t) = u(3,t) = 0$$

which are of the form $u(x,t) = X(x)T(t)$. The boundary conditions imply $X'(0) = 0$ and $X(3) = 0$.

$$2X(x)T'(t) = X''(x)T(t) \quad \Longrightarrow \quad \frac{X''(x)}{X(x)} = \frac{2T'(t)}{T(t)} = \text{const} =: -\lambda$$

In particular, $X(x)$ is a solution to the eigenvalue problem

$$X'' + \lambda X = 0, \quad X'(0) = 0, \; X(3) = 0.$$

By the previous problem, $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $X(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$, with $n = 0, 1, 2, \ldots$.

T solves $2T' + \lambda T = 0$, and hence, up to multiples, $T(t) = e^{-\frac{1}{2}\lambda t} = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}$.

Taken together, we have the solutions $u_n(x,t) = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}\cos\left(\frac{(2n+1)\pi}{6}x\right)$ solving $2u_t = u_{xx}$ and $u_x(0,t) = u(3,t) = 0$.

Note that $u_n(x,0) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$. In particular, the superposition

$$u(x,t) = 2u_1(x,t) + 7u_4(x,t) = 2e^{-\frac{1}{8}\pi^2 t}\cos\left(\frac{\pi x}{2}\right) + 7e^{-\frac{9}{8}\pi^2 t}\cos\left(\frac{3\pi x}{2}\right)$$

solves our heat conduction problem.

[It is not obvious that *every* initial temperature distribution $f(x)$ can be written as a (infinite) superposition of the $u_n(x,0)$. However, such "eigenfunction expansions" are always possible (thus extending what we know about ordinary Fourier series). See Chapter 10 in case you are interested to learn more.] □

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Problem 6. Using the Laplace transform, solve the initial value problem $x'' + 4x' + 4x = f(t)$ with $x(0) = 0$, $x'(0) = 0$ and

$$f(t) = \begin{cases} 2, & \text{for } 0 \leqslant t < 2, \\ t, & \text{for } 2 \leqslant t < 3, \\ 1, & \text{for } t \geqslant 3. \end{cases}$$

Solution. Let $X(s)$ be the Laplace transform of $x(t)$. Note that

$$f(t) = 2(u_0(t) - u_2(t)) + t(u_2(t) - u_3(t)) + u_3(t) = 2 + u_2(t)(t-2) - u_3(t)((t-3)+2).$$

An application of the Laplace transform yields

$$s^2 X + 4sX + 4X = \frac{2}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - 2\frac{e^{-3s}}{s}.$$

It follows that

$$X(s) = 2(1 - e^{-3s})\frac{1}{s(s+2)^2} + (e^{-2s} - e^{-3s})\frac{1}{s^2(s+2)^2}.$$

Using partial fractions, we have

$$\begin{aligned} X(s) &= (1 - e^{-3s})\frac{1}{2}\left(\frac{1}{s} - \frac{2}{(s+2)^2} - \frac{1}{s+2}\right) + (e^{-2s} - e^{-3s})\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+2)^2} + \frac{1}{s+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{2}{(s+2)^2} - \frac{1}{s+2}\right) + \frac{e^{-2s}}{4}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+2)^2} + \frac{1}{s+2}\right) - \frac{e^{-3s}}{4}\left(\frac{1}{s^2} + \frac{1}{s} - \frac{3}{(s+2)^2} - \frac{1}{s+2}\right). \end{aligned}$$

Taking the inverse transform, we obtain

$$\begin{aligned} x(t) &= \frac{1}{2}(1 - 2te^{-2t} - e^{-2t}) + \frac{u_2(t)}{4}\left((t-2) - 1 + (t-2)e^{-2(t-2)} + e^{-2(t-2)}\right) \\ &\quad - \frac{u_3(t)}{4}\left((t-3) + 1 - 3(t-3)e^{-2(t-3)} - e^{-2(t-3)}\right) \\ &= \frac{1}{2}(1 - (2t+1)e^{-2t}) + \frac{u_2(t)}{4}\left(t - 3 + (t-1)e^{-2(t-2)}\right) - \frac{u_3(t)}{4}\left(t - 2 - (3t-8)e^{-2(t-3)}\right). \end{aligned}$$

□

Finally, here is the table for the Laplace transform, which you will be given for the final exam.

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-sa}F(s)$

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