Very basic examples of differential equations

Example 1. If $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$.

We say that $y(x)=e^{x^2}$ is a solution to the differential equation (DE) $y'=2xy$. $\diamondsuit$

Example 2. If $y(x)=\sin(3x)$ then $y'(x)=3\cos(3x)=3\sqrt{1-\sin^2(3x)}$. Hence, $y(x)$ solves the differential equation $y'3\sqrt{1-y(x)^2}$.

On the other hand, $y''(x)=-9\sin(3x)=-9y(x)$. Thus, $y(x)$ also solves the second order differential equation $y''=-9y$. $\diamondsuit$

Example 3. Verify that $e^y y'=1$ is solved by $y(x)=\ln(x+C)$.

Solution. $y'(x)=\frac{1}{x+C}$ and $e^{y(x)}=x+C$. Hence, $e^y y'=1$ indeed.

This means that $y(x)=\ln(x+C)$ is a one-parameter family of solutions to the DE. $\diamondsuit$

Example 4. Consider the DE $y''=y'+6y$. For which r is e^{rx} a solution?

Solution. Plugging into the DE, we get $r^2e^{rx}=re^{rx}+6e^{rx}$ which we simplify to $r^2=r+6$. This has the two solutions $r=-2$, $r=3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x}+Be^{3x}$ is a two-parameter family of solutions to the DE. (It is no coincidence that the order of the DE is 2, whereas the previous example is order 1.) $\diamondsuit$

Example 5. Solve the DE $y'=x^2+x$.

Solution. $y(x)=\int(x^2+x)\mathrm{d}x=\frac{1}{3}x^3+\frac{1}{2}x^2+C$. By the way, from Calculus, we know that there are no other solutions. In other words, we found the general solution.

To single out a particular solution, we can impose initial values such as $y(0)=1$.

Solve $y'=x^2+x$ and $y(0)=1$. This is called an initial value problem (IVP).

Solution. $\left[\frac{1}{3}x^3+\frac{1}{2}x^2+C\right]_{x=0}=C\overset{!}{=}1$. Hence, $y(x)=\frac{1}{3}x^3+\frac{1}{2}x^2+1$ is the (unique) solution of the IVP. $\diamondsuit$

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