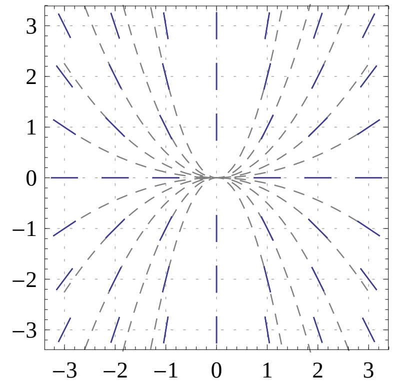
Review. Existence and uniqueness of solutions ◇

Example 15. Discuss the IVP $xy' = 2y$, $y(a) = b$.



Solution. First, write as $y' = f(x,y)$ with $f(x,y) = 2y/x$. We compute $\frac{\partial}{\partial y} f(x,y) = 2/x$.

Therefore, both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y) with $x \neq 0$. Hence, if $a \neq 0$ the IVP always has a (locally) unique solution.

We can verify that $y(x) = Cx^2$ solves the DE. (compare slope field!)

This means that the IVP with $y(0) = 0$ has infinitely many solutions. Since there are no other solutions (why?! look at slope field), the IVP with $y(0) = b$ has no solution if $b \neq 0$. ◇

Solving differential equations

Separation of variables

Example 16. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Write as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply with $\mathrm{d}x$ and integrate both sides (this might be nonsense but remember that we can [and should] test whether we found a solution, so let's not worry) to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$. Hence, $\ln|y| = 2\ln|x| + C$. Using $y(1) = 1$, we get $C = 0$ and thus $\ln(-y) = 2\ln x$ (close to the initial value, we have $|y| = -y$ and $|x| = x$). Solving for y, we find $y = -e^{2\ln x} = -x^2$. We easily verify that this is indeed a global solution (usually, it would only be a local solution and might have discontinuities such as poles). ◇

In general, separation of variables solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 17. $y' = x + y$ is a DE for which the variables can not be separated.

No worries, very soon we will have several tools to solve this DE as well. ◇

Example 18. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = k$. Integrating $\int \frac{1}{y^2}\mathrm{d}y = \int k\,\mathrm{d}x$, we find $-\frac{1}{y} = kx + C$ which we solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That's the solution we verified in an earlier lecture!

Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a singular solution because it is not part of the general solution (the one-parameter family found above). ◇

Remark 19. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us loose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$). ◇

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