Example 20. Solve $y' = ky$.

Solution. Write as $\frac{dy}{dx} = ky$, then $\frac{1}{y}dy = k dx$ (note that we just lost the solution $y = 0$).
Integrating gives $\ln|y| = kx + C$, hence $|y| = e^{kx+C}$. Since the RHS is never zero, $y = \pm e^{kx+C} = De^{kx}$ (with $D = \pm e^C$). Finally, note that $D = 0$ corresponds to the singular solution $y = 0$. In summary, the general solution is De^{kx} with $D \in \mathbb{R}$. ◇

Example 21. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$. Last time: unique solution guaranteed *a priori*.

Solution. Separate variables to get $y dy = -x dx$. Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, hence $x^2 + y^2 = D$ (with $D = 2C$). Using $y(0) = -3$, we find $0^2 + (-3)^2 = D$. Thus, $x^2 + y^2 = 9$ is an implicit form of the solution. In this case, we can solve for y to get $y = -\sqrt{9 - x^2}$. ◇

Linear first-order equations

Example 22. Solve $\frac{dy}{dx} = 2xy^2$.

Solution. $\frac{1}{y^2}\frac{dy}{dx} = 2x$, $-\frac{1}{y} = x^2 + C$. Hence the general solution is $y = \frac{1}{D - x^2}$. There is also the singular solution $y = 0$.

Solution. Note that $\frac{1}{y^2}\frac{dy}{dx} = 2x$ can be written as $\frac{d}{dx}\left[-\frac{1}{y}\right] = \frac{d}{dx}[x^2]$. Hence, $-\frac{1}{y} = x^2 + C$.

We now use the idea behind the second solution to solve other DEs.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with the so-called "integrating factor". ◇

Example 23. Solve $y' = x - y$. (Note that we cannot use separation of variables.)

Solution. $y' + y = x$, now multiply with e^x (we will see in a moment, how to find this "integrating factor").
Then $e^x y' + e^x y = \frac{d}{dx}[e^x y]$. On the other hand, $xe^x = \frac{d}{dx}[xe^x - e^x]$.
$\frac{d}{dx}[e^x y] = \frac{d}{dx}[xe^x - e^x]$ is equivalent to $e^x y = xe^x - e^x + C$. Hence, $y = x - 1 + Ce^{-x}$. ◇

In general, we can solve any linear first-order equation $y' + P(x)y = Q(x)$ in this way.

- We want to multiply with an integrating factor $f(x)$ such that the LHS of the DE becomes
$$f(x)y' + f(x)P(x)y = \frac{d}{dx}[f(x)y].$$
Since $\frac{d}{dx}[f(x)y] = f(x)y' + f'(x)y$, we need $f'(x) = f(x)P(x)$ for that.
- An $f(x)$ with that property is $f(x) = e^{\int P(x)dx}$. (Check!)
- The RHS of the DE only depends on x. It can be written as $f(x)Q(x) = \frac{d}{dx}[\int f(x)Q(x)dx]$.
- Hence, another way to write the DE is $\frac{d}{dx}[f(x)y] = \frac{d}{dx}[\int f(x)Q(x)dx]$.
- This shows that $f(x)y = \int f(x)Q(x)dx + C$, which means we have found the general solution (only need to divide by $f(x)$).
- Note that this solution exists on any interval on which P and Q are continuous. (This is better than what Theorem 11 can predict.)

Example 24. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. Write as $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$. Integrating factor $f(x) = e^{\ln x} = x$ (why do we write $\ln x$ instead of $\ln|x|$?). Hence, $xy = \int \left(\frac{1}{x} + 2\right)dx = \ln x + 2x + C$. Using $y(1) = 3$, we find $C = 1$. Solution $y = \frac{\ln(x) + 2x + 1}{x}$. ◇

Armin Straub
astraub@illinois.edu